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The CWI is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

CWI Tract 16

Mixed elliptic-hyperbolic partial differential operators: a case-study in Fourier integral operators

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Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

1980 Mathematics Subject Classification: 35A08, 35C05, 35M05, 42B20.
ISBN 90 6196 287 0

Printed in the Netherlands

ACKNOWLEDGEMENTS

This monograph is the result of research done at the Department of Mathematics and Computer Science of the Free University in Amsterdam during the last six years.

First of all, I would like to thank Prof.dr. G.Y. Nieuwland for his constant encouragement and his help and advice in many matters. I thank Prof.dr. J.J. Duistermaat and Prof.dr. E.M. de Jager for the interesting conversations I had with them.

To the National Aerospace Laboratory NLR in Amsterdam I am indebted for their support in the final stage of writing this monograph. All figures were made at the NLR. I thank Mr. K. Snel for typing the manuscript quickly and expertly.

I thank the Centre for Mathematics and Computer Science (CWI) in Amsterdam for publishing this monograph in their series CWI Tracts.

Finally I would like to thank everyone who has given me in any way moral support during these six years. Without them this piece of work would not have been completed.

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CHAPTER 1

INTRODUCTION

This piece of work has been motivated by a combined interest in both partial differential equations of mixed elliptic-hyperbolic type and Fourier Integral Operators. We test the utility of these operators for the study of the Tricomi operator $\partial^2/\partial t^2 + t\Delta_x$ and the operator $t(\partial^2/\partial t^2) + \Delta_x + \alpha(\partial/\partial t)$, which we will call the Pseudo Tricomi operator.

The Tricomi operator dates back to 1923, when Tricomi considered a simple boundary value problem for the equation $\partial^2 u/\partial t^2 + t(\partial^2 u/\partial x^2) = 0$ in $\mathbb{R}^2$ (Tricomi [26]). Then, in the forties it was Frankl' who saw the general connection between this equation and the theory of plane transonic gas flows (Frankl' [9]). This discovery aroused the interest in Tricomi and related operators. Some names that should be mentioned here are those of Bizadse, Gellerstedt, Germain and Bader, Morawetz, Protter. For more information see Bers [1]. Recent contributions are those of Schneider [22] and Gramtchev [12].
The Pseudo Tricomi operator we encountered first in Karol [17], in which boundary value problems were discussed for the associated homogeneous equation in $\mathbb{R}^2$. Although most probably this operator is not connected with problems of any practical interest, still it is very interesting, because it is of mixed type and characteristic along the "parabolic line" $t = 0$.

Most existing proofs for these problems use either integral equations or the energy integral method.
In the first case the region Ω in which the equation is considered, is divided into two regions Ω_1 and Ω_2, so that the equation is elliptic in Ω_1 and hyperbolic in Ω_2. First assuming $\tau(x) = u(x,0)$ to be known on the common boundary of Ω_1 and Ω_2, the problems in Ω_1 and Ω_2 are solved, giving then a (singular) integral equation for τ. Disadvantages of this method

are: 1. no closed formula for the solution is obtained and 2. it is not clear whether one obtains a solution in Ω or in $\Omega_1 \cup \Omega_2$. See for example Tricomi [26] and Bizadse [2].
For the energy integral method one has to find a Hilbert space H, a subspace S and an (energy) estimate $\|w\| \leq C\|{}^tLw\|$, for $w \in S$, L denoting the partial differential operator. The Hahn-Banach theorem then gives existence of a weak solution. The disadvantage of this method is that it is purely existential. See for example Morawetz [20] and Schneider [22].

Neither of these methods makes clear what happens in the transition area, when the operator changes type from hyperbolic to elliptic. In particular, the difference in bicharacteristic structure of Tricomi and Pseudo Tricomi operator is not represented. This becomes all too clear when we allow distributional data. Therefore it is natural to apply Fourier Integral Operators (FIOs). In the two publications Hörmander [15] and Duistermaat/Hörmander [7] it was shown that for certain PDOs P satisfying conditions on their bicharacteristic structure, FIOs can be used as the main tool for describing the singularities of solutions of the equation Pu = f in terms of the singularities of f. We might say they are employed to describe the singular part of solutions of the equation. Similar results for boundary value problems can also be obtained.

The Tricomi operator belongs to the class of operators satisfying the conditions referred to above, but the Pseudo Tricomi operator does not. One of the objectives of this research was to examine to what extent FIO-techniques are usable for the study of this operator.

FIO-techniques usually give solutions modulo smooth functions. This is probably one of the reasons why up to now they have been considered to be of mainly theoretical interest. Of course, in order to be of practical interest they should lead to 'exact' solutions. This might be arranged in three ways:

1. the classical way: first apply FIO-techniques to split off the singular part of the problem and solve the remaining problem, involving only smooth data, with other methods such as integral equations, numerical techniques, etc.
2. be more careful in the analysis: e.g., try to make asymptotic results exact, avoid cut-off functions, etc.
3. try to obtain estimates for the remainder.

A second objective of this research was to develop techniques in order to obtain exact solutions by way of method 2 or 3 for some feasible problems.

The organization of the next chapters is as follows.
In chapter 2 we give a summary of the theory of distributions, PDOs and FIOs that is used in the other chapters. This chapter is rather extensive in order to keep this work as much self-contained as possible and in order to explain at least the most simple facts about FIOs and their use for those who are not acquainted with the theory for these operators.
The chapters 3 and 4 deal with the Tricomi operator (in chapter 3 in $\mathbb{R}^{n+1}$). Fundamental solutions are constructed and analysed and boundary value problems with distributional data are discussed.
In chapter 5 we discuss an operator which is not of real principal type, that is, it does not have the properties indicated above.
We conclude with appendices in which some theory connected with Bessel and Airy functions is summarized and many technical lemmas used in the preceeding chapters are stated and proved.

The new results of this study are the following.
From the view-point of equations of mixed type: we solved boundary value problems with distributional data and we obtained explicit formulas for fundamental solutions. The formulas are closed in the transition area. The behaviour of solutions near the 'parabolic line' is described.
From the view-point of FIOs: most problems considered in this study have been solved exactly by applying adapted FIO-techniques.

These results could be obtained only by developing a large technical machinery. Fortunately, for the operators in question we were able to make use of various classical results, such as integral representations and asymptotic expansions for solutions of ordinary differential equations. In fact, it is an arguable point whether this machinery is so much larger than it is in the case of integral equations or energy integral methods, while the results are more detailed and more perspicuous. Therefore there seems to be no reason to shun FIO-methods in problems of practical interest, notably in mathematical physics.

CHAPTER 2

PRELIMINARIES

In this chapter we will give a summary of that part of the theory of distributions, Fourier Integral Operators (FIOs) and Partial Differential Equations (PDEs) which is relevant for the next chapters. Also the notation used will be exposed. Proofs of theorems will not be given here. Most of them can be found in Schwartz [23] and Hörmander [16]. Also in these books the most elementary theory of distributions can be found, which we assume to be wellknown.
Sections that are not more or less introductory of character but that contain results which are used only once or twice are marked by an asterisk (*).

2.1. Distribution spaces.

In this section we give a list of some more or less wellknown distribution spaces together with some defining properties. Let $\Omega \subset \mathbb{R}^n$ be an open subset and let $x = (x_1,\dots,x_n)$ denote a point of Ω.

$C_0^\infty(\Omega)$:= the set of smooth functions with compact support contained in Ω.

$\varphi_j \to 0$ in $C_0^\infty(\Omega)$ means that for some compact $K \subset \Omega$, supp $\varphi_j \subset K$ for all j and for every fixed multi-index α

$$\sup_{x\in K} |D^\alpha \varphi_j(x)| \to 0.$$

$\mathcal{D}'(\Omega)$:= the set of distributions on Ω.

It is the dual of $C_0^\infty(\Omega)$, that is, it is the set of linear forms u on $C_0^\infty(\Omega)$ so that $\varphi_j \to 0$ in $C_0^\infty(\Omega)$ implies $u(\varphi_j) \to 0$.

$C^\infty(\Omega)$:= the set of smooth functions on Ω.

$\varphi_j \to 0$ in $C^\infty(\Omega)$ means that

$$\sup_{x\in K} |D^\alpha \varphi_j| \to 0$$

for every multi-index α and every $K \subset \Omega$, K compact.

$E'(\Omega)$:= the set of distributions with compact support contained in Ω.

It is the dual of $C^\infty(\Omega)$.
Let now $\Omega = \mathbb{R}^n$.

$S(\mathbb{R}^n)$:= the set of smooth functions φ such that for every (α,β):

$$\sup |x^\beta D^\alpha \varphi(x)| < \infty.$$

$\varphi_j \to 0$ in $S(\mathbb{R}^n)$ means that for every (α,β) $\sup |x^\beta D^\alpha \varphi_j| \to 0$.

$S'(\mathbb{R}^n)$:= the set of temperate distributions on $\mathbb{R}^n$.

It is the dual of $S(\mathbb{R}^n)$.

$O_M(\mathbb{R}^n)$:= the set of smooth functions φ on $\mathbb{R}^n$ so that $f(x)D^\alpha\varphi(x)$ is bounded on $\mathbb{R}^n$ for every $f \in S(\mathbb{R}^n)$ and every multi-index α.

Convergence of $(\varphi_j) \subset O_M$ to zero means that $(fD^\alpha\varphi_j)$ converges uniformly to zero on $\mathbb{R}^n$ for every $f \in S$ and every α.

$O_C'(\mathbb{R}^n)$:= the set of rapidly decreasing distributions in S'.

Here u is said to be rapidly decreasing if for every $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have $\varphi * u \in S(\mathbb{R}^n)$.
Convergence of $(u_j) \subset O_C'$ to zero means that $(\varphi * x^\alpha u_j)$ converges uniformly to zero on $\mathbb{R}^n$ for every $\varphi \in C_0^\infty$ and every α.

The last two spaces are discussed in Schwartz [23].

REMARK.

1. O_C' is not the dual of O_M.
2. If $u \in O_C' \cap C^\infty$ then not necessarily $u \in S$. Take for example $u = e^{-x}\sin e^x$.
3. $E'(\mathbb{R}^n) \subset O_C'(\mathbb{R}^n) \subset S'(\mathbb{R}^n) \subset D'(\mathbb{R}^n)$.

$E_0(\mathcal{D}'_0)$:= the set of all $\varphi \in C^\infty(\mathbb{R}^+)$ $(u \in \mathcal{D}'(\mathbb{R}^+))$ so that for some $\varepsilon > 0$ $\varphi(u)$ is zero for $x < \varepsilon$.

$E_\infty(\mathcal{D}'_\infty)$:= the set of all $\varphi \in C^\infty(\mathbb{R}^+)$ $(u \in \mathcal{D}'(\mathbb{R}^+))$ so that for some $R < \infty$ $\varphi(u)$ is zero for $x > R$.

We say that $\varphi_j \to 0$ in E_0 ($u_j \to 0$ in $\mathcal{D}'_0$) if $\varphi_j \to 0$ in $C^\infty(\mathbb{R}^+)$ ($u_j \to 0$ in $\mathcal{D}'(\mathbb{R}^+)$) and ε can be chosen independently of j.
We say that $\varphi_j \to 0$ in E_∞ ($u_j \to 0$ in $\mathcal{D}'_\infty$) if $\varphi_j \to 0$ in $C^\infty(\mathbb{R}^+)$ ($u_j \to 0$ in $\mathcal{D}'(\mathbb{R}^+)$) and R can be chosen independently of j.
Any distribution in $\mathcal{D}'_0$ can be extended to a continuous linear form on E_∞, a distribution in $\mathcal{D}'_\infty$ can be extended to a continuous linear form on E_0.

It is clear that an element of E_0 or $\mathcal{D}'_0$ can be extended to $\mathbb{R}$ by defining it to be zero for $x \le 0$. This defines a continuous map between E_0 and $C^\infty(\mathbb{R})$ and between $\mathcal{D}'_0$ and $\mathcal{D}'(\mathbb{R})$.

Finally, $\mathcal{D}'_+$:= the set of distributions in $\mathcal{D}'(\mathbb{R})$ with support in $\overline{\mathbb{R}^+}$.

2.2. Convolution.

If $u_1 \in \mathcal{D}'(\mathbb{R}^n)$, $u_2 \in \mathcal{D}'(\mathbb{R}^n)$ and at least one in E', then $u_1 * u_2$ is the distribution so that for $\varphi \in C_0^\infty(\mathbb{R}^n)$:

$$\langle u_1 * u_2, \varphi\rangle = \langle u_1 \otimes u_2, \varphi(x+y)\rangle.$$

$*$ defines a separately continuous bilinear map.
If both u_1 and u_2 belong to E' or if $u_1 \in E'$ remains within a fixed compact set, then it is even continuous.
If all but at most one have compact support, the convolution of two or more distributions is welldefined, commutative and associative.
If $u_1 \in \mathcal{D}'$ and $u_2 \in C_0^\infty$ or $u_1 \in E'$ and $u_2 \in C^\infty$ then $u_1 * u_2$ is smooth and $(u_1 * u_2)(x) = \langle u_1(y), u_2(x-y)\rangle$.
Finally we have $\mathrm{supp}(u_1 * u_2) \subset \mathrm{supp}(u_1) + \mathrm{supp}(u_2)$.

This is the best known case. We will now discuss two more situations in which convolution can be defined.

If $u_1 \in \mathcal{D}'_+$ and $u_2 \in \mathcal{D}'_+$ then $u_1 * u_2$ is welldefined and $\mathrm{supp}(u_1 * u_2) \subset \mathrm{supp}(u_1) + \mathrm{supp}(u_2)$. The convolution of distributions in $\mathcal{D}'_+$ is commutative and associative. $*$ defines a continuous bilinear map.

If $u_1 \in \mathcal{E}'(\mathbb{R}^n)$ and $u_2 \in \mathcal{S}'(\mathbb{R}^n)$ then $u_1 * u_2 \in \mathcal{S}'(\mathbb{R}^n)$. This can be extended by continuity to the case $u_1 \in \mathcal{O}_C'$. Then $*$ defines a separately continuous bilinear map. If both distributions belong to $\mathcal{O}_C'$ it is even continuous. Then $u_1 * u_2 \in \mathcal{O}_C'$ as well. Again convolution is commutative and in case of several distributions in $\mathcal{S}'$, it is associative provided all but at most one belong to $\mathcal{O}_C'$.

REMARK. In more general cases convolution can be defined. However, this product is not necessarily associative. It is in the cases mentioned above.

2.3. Fourier transformation.

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ the Fourier transform $\hat{\varphi}$ of φ is given by:

$$\hat{\varphi}(\xi) = \int e^{-i\langle x,\xi\rangle}\varphi(x)dx.$$

Here $\xi \in \mathbb{R}^n$ and $\langle x,\xi\rangle := \Sigma_{j=1}^n x_j\xi_j$.

The inverse Fourier transform $\check{\varphi}$ is given by:

$$\check{\varphi}(\xi) = \frac{1}{(2\pi)^n}\int e^{i\langle x,\xi\rangle}\varphi(x)dx.$$

Both $\varphi \to \hat{\varphi}$ and $\varphi \to \check{\varphi}$ define continuous linear maps between $\mathcal{S}$ and itself. We have Fourier's inversion formula:

$$(\hat{\varphi})^\vee = \varphi \text{ and } (\check{\varphi})^\wedge = \varphi.$$

By duality $\hat{u}$ and $\check{u}$ can be defined for $u \in \mathcal{S}'(\mathbb{R}^n)$ as well.

$$\langle\hat{u},\varphi\rangle := \langle u,\hat{\varphi}\rangle \text{ and } \langle\check{u},\varphi\rangle := \langle u,\check{\varphi}\rangle.$$

Fourier's inversion formula also holds for $u \in \mathcal{S}'$. We also have:
Fourier transformation is an isomorfism between $\mathcal{O}_C'(\mathbb{R}^n)$ and $\mathcal{O}_M(\mathbb{R}^n)$.
See Schwartz [23].
As to the relation between Fourier transformation and convolution:
If $u_1 \in \mathcal{O}_C'$ and $u_2 \in \mathcal{S}'$, then $(u_1 * u_2)^\wedge = \hat{u}_1\cdot\hat{u}_2$ and $(u_1 * u_2)^\vee = (2\pi)^n\check{u}_1\cdot\check{u}_2$.
Here multiplication is welldefined because $\hat{u}_1 \in \mathcal{O}_M$ and $\check{u}_1 \in \mathcal{O}_M$.
Finally we mention the partial Fourier transforms $\tilde{\varphi}$ and $\tilde{\check{\varphi}}$ of $\varphi \in \mathcal{S}$ defined by:

$$\tilde{\varphi}(\xi',x_n) = \int e^{-i\langle x',\xi'\rangle}\varphi(x',x_n)dx',$$

$$\tilde{\check{\varphi}}(\xi',x_n) = \frac{1}{(2\pi)^{n-1}}\int e^{i\langle x',\xi'\rangle}\varphi(x',x_n)dx'.$$

Here $\xi' \in \mathbb{R}^{n-1}$ and $x = (x',x_n)$ with $x' \in \mathbb{R}^{n-1}$. Again $\varphi \to \tilde{\varphi}$ and $\varphi \to \hat{\varphi}$ define continuous linear maps between S and S. Further $(\tilde{\varphi})^{\frown} = \varphi = (\hat{\varphi})^{\sim}$.

2.4. Singular support and wave front set.

For $\xi \in \mathbb{R}^n$ we define $|\xi| := (\Sigma_{k=1}^n \xi_k^2)^{\frac{1}{2}}$. Let $f = f(\xi)$ be a function defined at least for $|\xi| \geq R$ for some $R < \infty$. f is said to be rapidly decreasing if

$$(2.4.1) \qquad \forall\, N\colon \exists\, C_N < \infty\colon |f(\xi)| \leq C_N(1+|\xi|)^{-N} \quad \text{for } |\xi| \geq R.$$

If $0 \neq \xi_0 \in \mathbb{R}^n$ then a conic neighbourhood U of ξ_0 is defined to be a neighbourhood of ξ_0 which is invariant under multiplication by positive scalars. f is said to be rapidly decreasing in the direction ξ_0 if for some conic neighbourhood U of ξ_0, condition (2.4.1) holds for $|\xi| \geq R$, $\xi \in U$.

If $u \in E'(\mathbb{R}^n)$ then it is wellknown that:
$u \in C_0^\infty \Leftrightarrow \hat{u}$ is rapidly decreasing.
More generally the singular support of a distribution $u \in \mathcal{D}'(\Omega)$, $\Omega \subset \mathbb{R}^n$, denoted by sing supp (u), is defined as follows:

$$(2.4.2) \qquad x_0 \in \text{sing supp}\,(u) \Leftrightarrow \forall\, \varphi \in C_0^\infty(\Omega) \text{ with } \varphi(x_0) = 1\colon \varphi u \notin C^\infty.$$

That is: $\widehat{\varphi u}$ is not rapidly decreasing.
Sing supp (u) is the smallest set outside of which u is smooth. Note that definition (2.4.2) does not take into account the directions in which $\widehat{\varphi u}$ is (is not) rapidly decreasing. For that purpose the wave front set of u, denoted by WF(u) is introduced. Let $\Omega \subset \mathbb{R}^n$ be open, $u \in \mathcal{D}'(\Omega)$, $x_0 \in \Omega$ and $\xi_0 \in \mathbb{R}^n$, $\xi_0 \neq 0$. Then:

$$(x_0,\xi_0) \in WF(u) \Leftrightarrow \forall\, \varphi \in C_0^\infty(\Omega) \text{ with } \varphi(x_0) = 1\colon \widehat{\varphi u}(\xi) \text{ is not rapidly decreasing in the direction } \xi_0.$$

u is smooth in (x,ξ) will mean $(x,\xi) \notin WF(u)$.

EXAMPLES.

1. Let $x = (x',x'') \in \mathbb{R}^{n+m}$, $\delta_{(x''=0)} \in \mathcal{D}'(\mathbb{R}^{n+m})$. Then for $\varphi \in C_0^\infty(\mathbb{R}^{n+m})$:

$$\widehat{\varphi\delta_{(x''=0)}}(\xi',\xi'') = \int e^{-i<x',\xi'>}\varphi(x',0)dx'.$$

Consider $x_0 = (x_0',x_0'')$, $\xi_0 = (\xi_0',\xi_0'')$. If $x_0'' \neq 0$ then for some φ: $\varphi(x_0) = 1$ and $\varphi(x',0) \equiv 0$. So $\widehat{\varphi\delta_{(x''=0)}} \equiv 0$ which implies $x_0 \notin$ sing supp $(\delta_{x''=0})$ and

$(x_0,\xi_0) \notin WF(\delta_{(x''=0)})$ for all ξ_0. If $x''_0 = 0$ then $\forall\, \varphi \in C_0^\infty$: $\varphi(x',0) \in C_0^\infty(\mathbb{R}^n)$. Provided $\varphi(x'_0,0) \neq 0$ then $0 \not\equiv \widehat{\varphi\delta_{(x''=0)}} = \widehat{(\varphi\delta)}(\xi')$ is rapidly decreasing in ξ'. But then easily follows: $\widehat{\varphi\delta}$ is rapidly decreasing in the direction of ξ_0 $\Leftrightarrow \xi'_0 \neq 0$.

So

$$\text{sing supp}\, \delta_{(x''=0)} = \{(x',0'') \mid x' \in \mathbb{R}^n\},$$

$$WF(\delta_{(x''=0)}) = \{(x',0'',0',\xi'') \mid x' \in \mathbb{R}^n,\ \xi'' \in \mathbb{R}^m \setminus 0\}.$$

2. Define $H \in \mathcal{D}'(\mathbb{R})$ by: $H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \phantom{\text{if }} x < 0 \end{cases}$.

$$\widehat{\varphi H}(\xi) = \int_0^\infty dx e^{-ix\xi}\varphi(x).$$

If $x_0 \neq 0$ then for some $\varphi \in C_0^\infty$: $\varphi(x_0) = 1$ and $H\varphi = \varphi$ or $H\varphi = 0$. In both cases $\widehat{\varphi H}$ is rapidly decreasing in ξ. If $x_0 = 0$ then

$$\widehat{\varphi H}(\xi) = \frac{\varphi(0)}{i\xi} + \frac{1}{i\xi}\int_0^\infty dx e^{-ix\xi}\frac{d\varphi}{dx} \quad \text{for } \xi \neq 0.$$

For $\varphi \in C_0^\infty$, $\varphi(0) = 1$ one more application of partial integration shows that this does not behave better than $\frac{1}{\xi}$ for $|\xi| \to \infty$. So

$$\text{sing supp}\,(H) = \{0\},$$

$$WF(H) = \{(0,\xi) \mid \xi \neq 0\}.$$

3. $u = \frac{1}{x+i0} \in \mathcal{D}'(\mathbb{R})$ is defined by $\langle u,\varphi\rangle = \lim_{\varepsilon\downarrow 0} \int \frac{\varphi(x)}{x+i\varepsilon}dx$. See also section 2.13. Then

$$\left(\varphi\cdot\frac{1}{x+i0}\right)^\wedge = \frac{1}{2\pi}\hat{\varphi} * \left(\frac{1}{x+i0}\right)^\wedge = -i\hat{\varphi} * H = -i\int_{-\infty}^{\xi}\hat{\varphi}(\eta)d\eta.$$

Clearly this is rapidly decreasing for $\xi \to -\infty$. If $\varphi(0) = 1$ then $\int_{-\infty}^{\infty}\hat{\varphi}(\eta)d\eta = 2\pi\varphi(0) = 2\pi$ so then $(\varphi\cdot\frac{1}{x+i0})^\wedge$ is not rapidly decreasing for $\xi \to +\infty$. If $x_0 \neq 0$ then for some $\varphi \in C_0^\infty$, $\varphi(x_0) = 1$ and $\varphi(x) = 0$ in a neighbourhood of the origin. So $\varphi\cdot\frac{1}{x+i0}$ is smooth. Therefore

$$\text{sing supp}\left(\frac{1}{x+i0}\right) = \{0\},$$

$$WF\left(\frac{1}{x+i0}\right) = \{(0,\xi) \mid \xi > 0\}.$$

<u>Properties of WF(u).</u>

Consider $\Omega \times \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ open, with coordinates (x,ξ).

1. Define $\pi_1 : \Omega \times \mathbb{R}^n \to \Omega$ by $\pi_1(x,\xi) = x$.
If $u \in \mathcal{D}'(\Omega)$ then $\pi_1[WF(u)] = \text{sing supp}\,(u)$.

2. If $u \in \mathcal{D}'(\Omega)$, $\varphi \in C^\infty(\Omega)$ then $WF(\varphi u) \subset WF(u)$. If α is a multi-index then $WF(D^\alpha u) \subset WF(u)$. So if P is a linear differential operator with smooth coefficients, then $WF(Pu) \subset WF(u)$.

3. If Ω_1 is an open subset of Ω then $u_1 := u|_{\Omega_1}$ is welldefined and $WF(u_1) = WF(u)|_{\Omega_1}$ where $WF(u)|_{\Omega_1} = WF(u) \cap \pi_1^{-1}(\Omega_1)$.

4. $WF(u)$ is a closed conic subset of $\Omega \times (\mathbb{R}^n \setminus 0)$. Here conic means that $(x,\xi) \in WF(u)$ and $\lambda > 0$ implies $(x,\lambda\xi) \in WF(u)$. If V is an arbitrary closed conic subset of $\Omega \times (\mathbb{R}^n \setminus 0)$ then there is a distribution $v \in \mathcal{D}'(\Omega)$ with $WF(v) = V$.

5. If $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^m$, $u_i \in \mathcal{D}'(\Omega_i)$, $i = 1,2$, then

$$
\begin{aligned}
WF(u_1 \otimes u_2) \subset\ & WF(u_1) \times WF(u_2) \cup \\
& (\mathrm{supp}(u_1) \times \{0\}) \times WF(u_2) \cup WF(u_1) \times (\mathrm{supp}(u_2) \times \{0\}).
\end{aligned}
$$

2.5. Convolution and singularities.

For $u \in \mathcal{E}'(\mathbb{R}^n)$, $v \in \mathcal{D}'(\mathbb{R}^n)$ the convolution $u * v$ is welldefined and

$$WF(u * v) \subset \{(x+y,\xi) \mid (x,\xi) \in WF(u) \text{ and } (y,\xi) \in WF(v)\}.$$

We will also encounter the following situation. Let $u \in \mathcal{E}'(\mathbb{R}^n_x)$, $v \in \mathcal{D}'(\mathbb{R}^n_x \times \mathbb{R}^m_y)$. Then convolution of u and v with respect to x only, written as $u *_x v$, is welldefined by:

$$u *_x v := (u \otimes \delta_{y=0}) * v$$

and

$$
\begin{aligned}
WF(u *_x v) \subset \{&(x_1 + x_2, y, \xi, \eta) \mid (x_2, y, \xi, \eta) \in WF(v) \\
&\text{and } [(x_1,\xi) \in WF(u) \text{ or } (\xi = 0 \text{ and } x_1 \in \mathrm{supp}(u))]\}.
\end{aligned}
$$

This will be proved in section A.7.

2.6. Operations with distributions.

The notion of the wave front set allows to generalize operations with functions to the case of distributions with suitable smoothness properties. This is done by continuous extension from the smooth case. For that a stronger type of convergence is used.

Let $\Omega \subset \mathbb{R}^n$ be open and Γ a closed cone in $\Omega \times (\mathbb{R}^n \setminus 0)$. So $(x,\xi) \in \Gamma$

implies $(x,\lambda\xi) \in \Gamma$ for all $\lambda > 0$.

$\mathcal{D}'_\Gamma(\Omega) := \{u \in \mathcal{D}'(\Omega) \mid WF(u) \subset \Gamma\}$.

Then $u \in \mathcal{D}'_\Gamma(\Omega) \Leftrightarrow$ for every $\varphi \in C_0^\infty(\Omega)$ and every closed cone $V \subset \mathbb{R}^n$ with $\Gamma \cap (\mathrm{supp}\,\varphi \times V) = \emptyset$ we have $\sup_V |\xi|^N |\widehat{\varphi u}(\xi)| < \infty$, $N = 1,2,3,\ldots$.

A sequence $(u_j) \subset \mathcal{D}'_\Gamma(\Omega)$ is said to converge to $u \in \mathcal{D}'_\Gamma(\Omega)$ if

i. $u_j \to u$ in $\mathcal{D}'(\Omega)$

ii. $\sup_V |\xi|^N |\widehat{\varphi u}(\xi) - \widehat{\varphi u}_j(\xi)| \to 0$, $j \to \infty$ for $N = 1,2,3,\ldots$ and (φ,V) as above.

$C_0^\infty(\Omega)$ is dense in $\mathcal{D}'_\Gamma(\Omega)$.

* Composition with smooth maps.

Let $\Omega_i \subset \mathbb{R}^{n_i}$, $i = 1,2$, be open sets. If f is a function $f : \Omega_1 \to \Omega_2$ and u is a function defined on Ω_2 then $u \circ f$ is a function defined on Ω_1.

If f is smooth and $Df(x)$ is surjective for every $x \in \Omega_1$ the map $u \to u \circ f$ defined for continuous functions can be extended in a unique way to a continuous lineair map $f^* : \mathcal{D}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$. f^*u is called the pullback of u by f.

The demand on surjectivity can be relaxed as follows:

(2.6.1) Let f be smooth and $N_f := \{(f(x),\eta) \mid {}^tf'(x)\eta = 0\}$. Then the pull-back f^*u can be defined in a unique way for all $u \in \mathcal{D}'(\Omega_2)$ with $N_f \cap WF(u) = \emptyset$ so that $f^*u = u \circ f$ when $u \in C^\infty$ and for any closed cone $\Gamma \subset \Omega_2 \times (\mathbb{R}^{n_2} \setminus 0)$ with $\Gamma \cap N_f = \emptyset$ we have a continuous map $f^* : \mathcal{D}'_\Gamma(\Omega_2) \to \mathcal{D}'_{f^*\Gamma}(\Omega_1)$. Here $f^*\Gamma = \{(x, {}^tf'(x)\eta) \mid (f(x),\eta) \in \Gamma\}$. Then $WF(f^*u) \subset f^*WF(u)$.

Note that if $Df(x)$ is surjective, $N_f = \{(f(x),0) \mid x \in \Omega_1\}$, so $N_f \cap WF(u) = \emptyset$ for all $u \in \mathcal{D}'(\Omega_2)$.
The set N_f is called the set of normals of the map f:

$$ {}^tf'(x)\eta = 0 \Leftrightarrow \forall\, \xi: \; 0 = \langle {}^tf'(x)\eta,\xi\rangle = \langle \eta, f'(x)\xi\rangle. $$

Therefore the condition $N_f \cap WF(u) = \emptyset$ says that WF(u) contains no points (y,η) which are orthogonal to the image of $T(\Omega_1)$, the tangent bundle of Ω_1, under the map $(x,\xi) \to (f(x),Df(x)\xi)$.

* Multiplication.

Multiplication of two distributions can be defined for instance if for every $x_0 \in \Omega$ one of the distributions is smooth in x_0. This can be generalized as follows:

(2.6.2) Multiplication of u and $v \in \mathcal{D}'(\Omega)$ can be defined if $(x,\xi) \in WF(u)$ implies $(x,-\xi) \notin WF(v)$. So if u is not smooth in (x,ξ), v must be smooth in $(x,-\xi)$.
uv is defined as the pullback of $u \otimes v$ by the diagonal map $x \to (x,x)$. Property 5 of section 2.4 and the result on composition with a smooth map given above provide the arguments. Then:

$$WF(uv) \subset \{(x,\xi+\eta) \mid (x,\xi) \in WF(u) \text{ or } \xi = 0,\ (x,\eta) \in WF(v) \text{ or } \eta = 0\}.$$

If $u \in \mathcal{E}'$, $v \in \mathcal{E}'$ and if we write strictly formally

$$\widehat{uv} = \frac{1}{(2\pi)^n} \hat{u} * \hat{v} = \frac{1}{(2\pi)^n} \int \hat{u}(\eta)\hat{v}(\xi-\eta)d\eta,$$

condition (2.6.2) gives an indication that this integral will be convergent.

Restrictions.

(2.6.3) Let Ω be an open subset of $\mathbb{R}^n$ and Y a submanifold of Ω with normal bundle denoted by N(Y). If $u \in \mathcal{D}'(\Omega)$ has $WF(u) \cap N(Y) = \emptyset$ then the restriction $u|_Y$ of u to Y can be defined as the pullback by the inclusion $Y \hookrightarrow \Omega$. This restriction is unique in the sense that it is sequentially continuous from $\mathcal{D}'_\Gamma(\Omega)$ to $\mathcal{D}'(Y)$ for every closed cone $\Gamma \subset \Omega \times (\mathbb{R}^n \setminus 0)$ with $\Gamma \cap N(Y) = \emptyset$.

We discuss this locally. So $\Omega = \mathbb{R}^{k+\ell}$, $Y = \{(x',x'') \mid x' \in \mathbb{R}^k,\ x'' = 0\}$. Then $N(Y) = \{(x',0,0,\xi'') \mid x' \in \mathbb{R}^k,\ \xi'' \in \mathbb{R}^\ell\}$. $\iota : x' \to (x',0)$ is the inclusion map. Then $N_\iota = \{(x',0,\eta) \mid {}^t\iota'(x')\eta = 0\} = N(Y)$. So the condition given in paragraph (2.6.1) is satisfied.

$$\text{If } \Omega = \mathbb{R}^2,\ Y = \mathbb{R} \times \{0\},\ u(x_1,x_2) = \begin{cases} x_2 & \text{for } x_2 > 0 \\ 0 & \text{for } x_2 < 0 \end{cases},$$

then it is easily seen that WF(u) = N(Y), so u does not fit into this framework. Of course $u|_Y$ can be defined here as well. For later reference we discuss one more extension of the notion of restriction that covers this case.

Let $\Omega \subset \mathbb{R}^n$ open, Y a (n-1)-dimensional smooth submanifold of Ω so that $\Omega \setminus Y$ consists of two components. The restriction of smooth functions on Ω to Y can be extended to a continuous map between $H^{\alpha}_{loc}(\Omega) \to H^{\alpha-\frac{1}{2}}_{loc}(Y)$ if $\alpha > \frac{1}{2}$. This is a wellknown theorem in the theory of Sobolev spaces. See for instance Lions/Magenes [18].

2.7. Kernels.

We start this section with the Schwartz kernel theorem. Let $\Omega_i \subset \mathbb{R}^{n_i}$ be open subsets, $i = 1,2$. If $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ then we get a map $\mathcal{K} : C_0^\infty(\Omega_2) \to \mathcal{D}'(\Omega_1)$ by:

(2.7.1) $\quad \langle \mathcal{K}\varphi, \psi\rangle := \langle K, \psi \otimes \varphi\rangle$.

Here $\varphi \in C_0^\infty(\Omega_2)$, $\psi \in C_0^\infty(\Omega_1)$ and $\psi \otimes \varphi$ is the usual tensor product.

For every $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ the prescription (2.7.1) defines a linear map $\mathcal{K}$ from $C_0^\infty(\Omega_2)$ to $\mathcal{D}'(\Omega_1)$. This map is continuous in the sense that $\varphi_j \to 0$ in $C_0^\infty(\Omega_2)$ implies $\mathcal{K}\varphi_j \to 0$ in $\mathcal{D}'(\Omega_1)$. Conversely, for every linear map $\mathcal{K}$ with these properties there is a unique $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ so that (2.7.1) is valid. K is called the kernel of $\mathcal{K}$.

If A and B are sets, $R \subset A \times B$ and $C \subset B$ then $R \circ C := \{a \in A \mid \exists\, c \in C: (a,c) \in R\}$.

(2.7.2) Then $\operatorname{supp}(\mathcal{K}\varphi) \subset \operatorname{supp}(K) \circ \operatorname{supp}(\varphi)$.

Two cases in which the map $\mathcal{K}$ can be extended to $\mathcal{E}'(\Omega_2)$ are:

1. If $K \subset C^\infty(\Omega_1 \times \Omega_2)$ then $\mathcal{K} : \mathcal{E}'(\Omega_2) \to C^\infty(\Omega_1)$ continuously and $(\mathcal{K}u)(x_1) = \langle u, K(x_1, \cdot)\rangle$. Conversely every continuous map with these properties is defined in this way by a kernel $K \subset C^\infty$. The corresponding operator is called smoothing.
2. Define ${}^t\mathcal{K}$, the transpose of $\mathcal{K}$ by $\langle {}^t\mathcal{K}\psi, \varphi\rangle := \langle \psi, \mathcal{K}\varphi\rangle$, $\varphi \in C_0^\infty(\Omega_2)$, $\psi \in C_0^\infty(\Omega_1)$. If ${}^t\mathcal{K} : C_0^\infty(\Omega_1) \to C^\infty(\Omega_2)$ continuously, then $\mathcal{K}$ can be extended to a continuous map between $\mathcal{E}'(\Omega_2)$ and $\mathcal{D}'(\Omega_1)$ by $\langle \mathcal{K}u, \psi\rangle := \langle u, {}^t\mathcal{K}\psi\rangle$.

Conditions for other extensions can be expressed in terms of the wave front set of the kernel K.

If $\varphi \in C_0^\infty(\Omega_2)$ $WF(\mathcal{K}\varphi)$ can be estimated in terms of $WF(K)$ as follows:

(2.7.3) $\quad WF(K\varphi) \subset \{(x_1,\xi_1) \mid (x_1,x_2,\xi_1,0) \in WF(K) \text{ for some } x_2 \in \operatorname{supp}(\varphi)\}.$

Define

$$WF(K)_{\Omega_1} := \{(x_1,\xi_1) \mid (x_1,x_2,\xi_1,0) \in WF(K) \text{ for some } x_2 \in \Omega_2\},$$
$$WF'(K)_{\Omega_2} := \{(x_2,\xi_2) \mid (x_1,x_2,0,-\xi_2) \in WF(K) \text{ for some } x_1 \in \Omega_1\}.$$

Note that

(2.7.4)
$$WF(K)_{\Omega_1} = \emptyset \text{ implies } K : C_0^\infty \to C^\infty.$$
$$WF'(K)_{\Omega_2} = \emptyset \text{ implies } {}^tK : C_0^\infty \to C^\infty,$$

for the kernel of ${}^tK$ is given by ${}^tK$ defined by ${}^tK(x_1,x_2) = K(x_2,x_1)$.

(2.7.5) For $u \in \mathcal{E}'(\Omega_2)$ with $WF(u) \cap WF'(K)_{\Omega_2} = \emptyset$, $Ku \in \mathcal{D}'(\Omega_1)$ can be defined: for such u Ku can be defined in a unique way so that the map $\mathcal{E}'(M) \cap \mathcal{D}'_\Gamma(\Omega_2) \ni u \to Ku \in \mathcal{D}'(\Omega_1)$ is continuous for all compact sets $M \subset \Omega_2$ and all closed conic sets Γ so that $\Gamma \cap WF'(K)_{\Omega_2} = \emptyset$. Here $\mathcal{E}'(M)$ is the set of distributions with support in M. The propagation of singularities is as follows:

(2.7.6) $\quad WF(Ku) \subset WF(K)_{\Omega_1} \cup WF'(K) \circ WF(u),$

where $WF'(K) = \{(x_1,x_2,\xi_1,\xi_2) \mid (x_1,x_2,\xi_1,-\xi_2) \in WF(K)\}$. Also $WF(K)$, which is a subset of $(\Omega_1 \times \Omega_2) \times \mathbb{R}^{n_1+n_2}$, is identified with the set $\{(x_1,\xi_1,x_2,\xi_2) \mid (x_1,x_2,\xi_1,\xi_2) \in WF(K)\}$, which is considered as a subset of $(\Omega_1 \times \mathbb{R}^{n_1}) \times (\Omega_2 \times \mathbb{R}^{n_2})$.

<u>REMARK</u>. If $WF'(K)_{\Omega_2} = \emptyset$ then ${}^tK : C_0^\infty \to C^\infty$. So K can be extended to $\mathcal{E}'(\Omega_2)$ by transposition. See case 2. This map coincides with the map defined in paragraph (2.7.5) for both maps are equal on $C_0^\infty(\Omega_2)$ and continuous.

If $\Omega_3 \subset \mathbb{R}^{n_3}$ open, $K_1 \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ and $K_2 \in \mathcal{D}'(\Omega_2 \times \Omega_3)$, we now discuss the composition $K_1 \circ K_2$.

(2.7.7) For $\chi \in C_0^\infty(\Omega_3)$ property (2.7.2) shows that a sufficient condition for $K_2\chi$ to be in $\mathcal{E}'(\Omega_2)$ is that the projection $\pi : \operatorname{supp} K_2 \ni (x_2,x_3) \to x_3$ is proper, that is $\pi^{-1}(M)$ is compact in $\operatorname{supp} K_2$ for every compact $M \subset \Omega_3$. Moreover, if $WF'(K_1)_{\Omega_2} \cap WF(K_2)_{\Omega_2} = \emptyset$, property (2.7.3) and condition (2.7.5) show that $K_1 \circ K_2$ is a continuous map between $C_0^\infty(\Omega_3)$ and $\mathcal{D}'(\Omega_1)$. So $K_1 \circ K_2$ has a kernel $K \in \mathcal{D}'(\Omega_1 \times \Omega_3)$. For the wave front set of K holds:

$$(2.7.8) \qquad WF'(K) \subset WF'(K_1) \circ WF'(K_2) \cup WF(K_1)_{\Omega_1} \times (\Omega_3 \times \{0\}) \cup (\Omega_1 \times \{0\}) \times WF'(K_2)_{\Omega_3}.$$

2.8. Oscillatory Integrals.

In this section we call attention to a very important class of operators which have kernels defined as oscillatory integrals. These operators are the local Fourier Integral Operators (FIOs). As in the previous sections, all definitions will be given locally, that is, we only consider the case that we are working on open subsets of some $\mathbb{R}^n$. In the next chapters, no global, invariant theory on manifolds is necessary. Therefore we leave it out in this chapter, too.

The materials for oscillatory integrals are phase functions and symbols.

Phase functions.

Let $\Omega \subset \mathbb{R}^n$ be open and $\Gamma \subset \Omega \times (\mathbb{R}^N \setminus 0)$ be an open cone for some N. So $x \in \Omega$, $(x,\theta) \in \Gamma$ implies $(x,\lambda\theta) \in \Gamma$ for every $\lambda > 0$.

Let $\varphi \in C^\infty(\Gamma)$ be a smooth function in Γ which satisfies:

i) φ is homogeneous of order 1 in θ, that is: $\varphi(x,\lambda\theta) = \lambda\varphi(x,\theta)$ if $(x,\theta) \in \Gamma$ and $\lambda > 0$.

ii) φ is real.

iii) $d\varphi \neq 0$ in Γ.

Then φ is called a real phase function in Γ.

In many situations condition ii) can be replaced by the condition $\operatorname{Im} \varphi \geq 0$. Most of the time we will work with real valued functions. So we omit the adjective real and assume φ real unless otherwise stated. If $\Gamma = \Omega \times (\mathbb{R}^N \setminus 0)$ we simply say φ is a phase function.

The set C_φ is defined as

$$C_\varphi := \{(x,\theta) \in \Gamma \mid \varphi_\theta(x,\theta) = 0\}.$$

Here $\varphi_\theta := (\partial\varphi/\partial\theta_1,\dots,\partial\varphi/\partial\theta_N)$. Also $\varphi_x := (\partial\varphi/\partial x_1,\dots,\partial\varphi/\partial x_n)$.

φ is called non-degenerate if

$$\forall\ (x,\theta) \in C_\varphi\text{: the differentials } d(\partial\varphi/\partial\theta_k),\ k=1,\dots,N \text{ are linear independent at } (x,\theta).$$

This implies that C_φ is a submanifold of dimension n in Γ.
The set Λ_φ is defined as

(2.8.1) $$\Lambda_\varphi := \{(x,\varphi_x(x,\theta)) \mid (x,\theta) \in C_\varphi\}.$$

The set Λ_φ will play an important role in the description of the wave front set of a kernel given as an oscillatory integral. If φ is non-degenerate it is an n-dimensional conic submanifold of $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ and the map $C_\varphi \ni (x,\theta) \to (x,\varphi_x(x,\theta)) \in \Lambda_\varphi$ defines a diffeomorphism of a conic open neighbourhood of an arbitrary point of C_φ onto a conic neighbourhood of the image of that point in Λ_φ.

REMARK. A first step towards a global discussion is to note that Λ_φ can be considered as a submanifold of $T^*(\Omega) \setminus 0$. Here $T^*(\Omega)$ denotes the cotangent bundle of Ω, which is the dual of the tangent bundle $T(\Omega)$ of Ω.
If in future we use the notation $T^*(\Omega)$, one can always interpret this locally as $\Omega \times \mathbb{R}^n$ with $n = \dim \Omega$. Then $T^*(\Omega) \setminus 0 \simeq \Omega \times (\mathbb{R}^n \setminus 0)$.

Symbols.

Let $\Omega \subset \mathbb{R}^n$ be open and let m, ρ and δ be real numbers with $0 < \rho \le 1$ and $0 \le \delta < 1$. Then $S^m_{\rho,\delta}(\Omega \times \mathbb{R}^N)$ is the set of all $s \in C^\infty(\Omega \times \mathbb{R}^N)$ so that for every compact set $K \subset \Omega$ and every multi-index α,β there is a constant $C_{K,\alpha,\beta} < \infty$ so that

(2.8.2) $$|D^\beta_x D^\alpha_\theta s(x,\theta)| \le C_{K,\alpha,\beta}(1+|\theta|)^{m-\rho|\alpha|+\delta|\beta|} \quad \text{for } x \in K,\ \theta \in \mathbb{R}^N.$$

So the estimate improves after differentiation with respect to θ and does not get too much worse after differentiation with respect to x.

The best possible constants in estimate (2.8.2) are semi-norms that turn $S^m_{\rho,\delta}$ into a Fréchet space. The elements of $S^m_{\rho,\delta}$ are called symbols of order m and type (ρ,δ). We also say that s satisfies $S^m_{\rho,\delta}$-estimates. If $s_1 \in S^{m_1}_{\rho,\delta}$ and $s_2 \in S^{m_2}_{\rho,\delta}$ then $s_1 s_2 \in S^{m_1+m_2}_{\rho,\delta}$.

If Γ is an open conic subset of $\Omega \times \mathbb{R}^N$, s is a function defined on Γ, smooth for $|\theta| > R$ for some R then s is said to satisfy $S^m_{\rho,\delta}$-estimates in Γ if s satisfies the estimates (2.8.2) for $x \in K$, $|\theta| > R$, $(x,\theta) \in \Gamma$.
If $\Gamma = \Omega \times \mathbb{R}^N$ we say that s satisfies $S^m_{\rho,\delta}$-estimates for large θ.

For $s \in S^m_{\rho,\delta}$: cone supp (s) := $\{(x,\lambda\theta) \mid (x,\theta) \in \mathrm{supp}(s),\ \lambda \ge 0\}$.

Finally, we say that s is rapidly decreasing in an open cone Γ if s satisfies $S^m_{\rho,\delta}$-estimates in Γ for every $m \in \mathbb{R}$.

EXAMPLES. A polynomial of degree m is in $S^m_{1,0}$. If $s \in C^\infty$ and homogeneous of degree m in θ for large θ then $s \in S^m_{1,0}$.

(2.8.3) A method for constructing symbols is:
If s is a symbol in $S^0_{\rho,\delta}(\Omega \times \mathbb{R}^N)$ and f is a C^∞-function in a neighbourhood of the limit points of s when $|\theta| \to \infty$ while x may vary, then $f(s(x,\theta))$ satisfies $S^0_{\rho,\delta}$-estimates for large θ.

Oscillatory Integrals.

Let φ be a phase function in the open cone $\Gamma \subset \Omega \times \mathbb{R}^N$. Let F be a closed cone in $\Gamma \cup (\Omega \times \{0\})$, s a symbol with $\mathrm{supp}(s) \subset F$. Let $u \in C_0^\infty(\Omega)$. Consider the expression $I_\varphi(su)$ given by

$$(2.8.4) \qquad I_\varphi(su) = \int e^{i\varphi(x,\theta)} s(x,\theta)u(x)dxd\theta.$$

If m is the order of s and $m + N < 0$ this is an absolutely convergent integral. This is also the case if s vanishes for large θ. In a unique way for all $s \in \bigcup_{m,\rho,\delta} S^m_{\rho,\delta}(\Omega \times \mathbb{R}^N)$ with support in F and all $u \in C_0^\infty(\Omega)$ an interpretation for this integral can be given so that $I_\varphi(su)$ is a continuous linear function of $s \in S^m_{\rho,\delta}$ for every fixed $u \in C_0^\infty(\Omega)$, $m \in \mathbb{R}$, $0 < \rho \le 1$ and $0 \le \delta < 1$. Then the linear form $u \to I_\varphi(su)$ is a distribution of order $\le k$ if $s \in S^m_{\rho,\delta}$ and $m - k\rho < -N$, $m - k(1-\delta) < -N$.

The proof is based on the method of stationary phase.
The properties of φ make it possible to construct a differential operator $L = \Sigma\, a_j(\partial/\partial\theta_j) + \Sigma\, b_k(\partial/\partial x_k) + c$, $a_j \in S^0_{1,0}$, $b_k, c \in S^{-1}_{1,0}$, so that $Le^{i\varphi} = e^{i\varphi}$. Then ${}^tL$ maps $S^m_{\rho,\delta}$ into $S^{m-\varepsilon}_{\rho,\delta}$ with $\varepsilon = \min(\rho, 1-\delta) > 0$. For s vanishing for large θ in formula (2.8.4) repeated application of partial integration with L gives the expression

$$I_\varphi(su) = \int e^{i\varphi(x,\theta)} ({}^tL)^k(su)dxd\theta.$$

If $m - k\varepsilon + N < 0$ this integral is absolutely convergent for all $s \in S^m_{\rho,\delta}$ and $I_\varphi(su)$ can be defined for such s by this expression. Since $\varepsilon > 0$, this is possible for every m,ρ,δ as above. It can be shown that $I_\varphi(su)$ has the properties given above.

The extended definition of expression (2.8.4) will be called an oscillatory integral. We will use the notation (2.8.4) for $I_\varphi(su)$ even if the integral is not absolutely convergent. For the distribution $u \to I_\varphi(su)$ the notation

$$\int d\theta e^{i\varphi(x,\theta)} s(x,\theta)$$

is used.

If φ and s depend continuously on a parameter $y \in \mathbb{R}^m$ in $C^\infty(\Gamma)$ and $S^m_{\rho,\delta}(\Omega \times \mathbb{R}^N)$ respectively, $\operatorname{supp}(s) \subset F$, then $I_\varphi(su)$ is a continuous function of y. This remark can also be used to justify differentiation with respect to y under the integral sign.

EXAMPLES.

1. For $u \in C_0^\infty(\mathbb{R}^n)$:

$$u = \check{\hat{u}} = \frac{1}{(2\pi)^n} \int d\theta e^{i\langle x,\theta\rangle} \int dy e^{-i\langle y,\theta\rangle} u(y).$$

If $\chi \in S$, $\chi(0) = 1$ then $\chi(\varepsilon\theta) \to 1$ in $S^m_{1,0}$ for $\varepsilon \downarrow 0$, $m > 0$. This is a simple exercise. Also $\chi(\varepsilon\theta)\hat{u}(\theta) \to \hat{u}(\theta)$ in S, so

$$u = \lim \frac{1}{(2\pi)^n} \int d\theta e^{i\langle x,\theta\rangle} \chi(\varepsilon\theta) \int dy e^{-i\langle y,\theta\rangle} u(y)$$

$$= \lim \frac{1}{(2\pi)^n} \int e^{i\langle x-y,\theta\rangle} \chi(\varepsilon\theta) u(y) dy d\theta = \frac{1}{(2\pi)^n} \int e^{i\langle x-y,\theta\rangle} u(y) dy d\theta.$$

The last expression is considered as an oscillatory integral with phase function $\langle x-y,\theta\rangle$, symbol $1/(2\pi)^n$, x considered as a parameter.

2. More generally, if $P = \Sigma_{|\alpha|\le m} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}$ is a partial differential operator (PDO) of order m, $a_\alpha \in C^\infty$, then

$$Pu = \frac{1}{(2\pi)^n} \int e^{i\langle x-y,\theta\rangle} p(x,\theta) u(y) dy d\theta.$$

$p(x,\theta) := \Sigma_{|\alpha|\le m} a_\alpha(x)(i\theta)^\alpha$ is called the symbol of P. So for fixed x, Pu can be considered as an oscillatory integral with phase function $\langle x-y,\theta\rangle$ and symbol $p(x,\theta)/(2\pi)^n \in S^m_{1,0}$. The associated distribution is

$$\int e^{i\langle x-y,\theta\rangle} p(x,\theta) d\theta = \sum_{|\alpha|\le m} a_\alpha(x)(-1)^{|\alpha|} \delta^{(\alpha)}_{(y=x)}.$$

3. Consider the Cauchy problem

$$\Delta u - \frac{\partial^2 u}{\partial t^2} = 0 \text{ in } \mathbb{R}^{n+1},$$

$$u = 0, \quad \frac{\partial u}{\partial t} = \delta_{(x=0)} \text{ on } t = 0.$$

Here $\Delta := \Sigma_{k=1}^{n} \partial^2/\partial x_k^2$. Partial Fourier transformation with respect to x gives the solution

$$u(x,t) = \frac{1}{(2\pi)^n} \int \left[e^{i(<x,\theta>+t|\theta|)} - e^{i(<x,\theta>-t|\theta|)}\right] \frac{d\theta}{2i|\theta|}.$$

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 in a neighbourhood of 0 and substitute $1 = \chi(\theta) + (1 - \chi(\theta))$. Then

$$u(x,t) - \frac{1}{(2\pi)^n} \int \left[e^{i(<x,\theta>+t|\theta|)} - e^{i(<x,\theta>-t|\theta|)}\right] \frac{\chi(\theta)d\theta}{2i|\theta|}$$

is the sum of two oscillatory integrals with phase functions $<x,\theta> \pm t|\theta|$ respectively and symbol $\frac{1}{(2\pi)^n}(1-\chi(\theta))\frac{1}{2i|\theta|} \in S^{-1}_{1,0}$.

Note that the integral involving $\chi(\theta)$ is absolutely convergent and defines a function which is smooth in (x,t). Therefore u can be said to be the sum of two oscillatory integrals modulo a smooth function. If one is interested only in the singularities of u this function can be neglected. On the other hand, note that since the integrand is clearly bounded by $|t|$, on compact subsets the contribution of this integral can be made arbitrarily small by choosing the support of χ small enough.

We will now discuss the singularities of the distribution $S : u \to I_\varphi(su)$. Let φ and s be as in the definition of $I_\varphi(su)$. Then:

$$WF(S) \subset \Lambda_\varphi.$$

Here Λ_φ is defined by equation (2.8.1).

(2.8.5) Moreover, if $(x_0,\xi_0) \in \Lambda_\varphi$ and s is rapidly decreasing in a conic neighbourhood of every (x_0,θ) so that $(x_0,\theta) \in C_\varphi$ and $\xi_0 = \lambda\varphi_x(x_0,\theta)$ for some $\lambda > 0$, then $(x_0,\xi_0) \notin WF(S)$.

The importance of the set C_φ seems obvious. As to the set Λ_φ, note that for $\psi \in C_0^\infty$:

$$(\psi S)^\wedge(\xi) = \iint e^{i(\varphi(x,\theta)-<x,\xi>)}\psi(x)s(x,\theta)dxd\theta.$$

Also

$$\frac{\partial}{\partial x}\left[\varphi(x,\theta) - <x,\xi>\right] = \varphi_x(x,\theta) - \xi = 0 \Leftrightarrow \xi = \varphi_x(x,\theta)!$$

The method referred to in the discussion of the integral (2.8.4) can then be applied in order to verify the fact that $WF(S) \subset \Lambda_\varphi$, using an operator $L = \Sigma b_j(\partial/\partial x_j) + c$ so that $b_j, c \in S^{-1}_{1,0}$ and $L(\varphi(x,\theta) - \langle x,\xi\rangle) = \varphi(x,\theta) - \langle x,\xi\rangle$ for $\xi \neq \varphi_x(x,\theta)$.

Let now $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ be open. The coordinates in Ω_1 and Ω_2 we now denote by x and y respectively. Let $\varphi = \varphi(x,y,\theta)$ be a phase function in an open cone $\Gamma \subset \Omega \times \mathbb{R}^N$, $s = s(x,y,\theta)$ a symbol with supp(s) in a closed cone contained in Γ. Then

$$K(x,y) = \int e^{i\varphi(x,y,\theta)} s(x,y,\theta)d\theta$$

can be considered as a distribution kernel with

$$WF(K) \subset \{(x,y,\varphi_x(x,y,\theta),\varphi_y(x,y,\theta)) \mid \varphi_\theta(x,y,\theta) = 0\}.$$

Then

$$WF(K)_{\Omega_1} \subset \{(x,\varphi_x) \mid \exists\ (x,y,\theta) : \varphi_\theta(x,y,\theta) = 0 \text{ and } \varphi_y(x,y,\theta) = 0\},$$

$$WF'(K)_{\Omega_2} \subset \{(y,-\varphi_y) \mid \exists\ (x,y,\theta) : \varphi_\theta(x,y,\theta) = 0 \text{ and } \varphi_x(x,y,\theta) = 0\}.$$

Note that $\varphi_\theta = 0$ and $\varphi_y = 0$ implies $\varphi_x \neq 0$, for φ is a phase function. In particular if φ has no critical points as function of (y,θ), $WF(K)_{\Omega_1} = \emptyset$ so the associated operator K maps $C_0^\infty(\Omega_2)$ to $C^\infty(\Omega_1)$. If φ has no critical points as function of (x,θ), K extends to an operator $\mathcal{E}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$. See section 2.7.

(2.8.6) An operator associated with a kernel defined as above, with a phase function φ satisfying the conditions that it does not have critical points with respect to (x,θ) and to (y,θ), is called a Fourier Integral Operator (FIO). In that case formula (2.7.6) shows that $WF(Ku) \subset WF'(K) \circ WF(u)$.

(2.8.7) The order μ of a FIO is defined as: $\mu = m + \frac{1}{2}N - \frac{1}{4}(n_1 + n_2)$, m is the order of s, $s \in S^m(\Omega \times \mathbb{R}^N)$, $n_1 = \dim \Omega_1$ and $n_2 = \dim \Omega_2$. This definition may seem to appear out of the blue, but it has its grounds in the composition rules for FIOs.

(2.8.8) An operator $C_0^\infty \to \mathcal{D}'$ is said to be properly supported if the kernel K of this operator satisfies:

the projections $\text{supp}(K) \ni (x,y) \to x \in \Omega_1$,
$\text{supp}(K) \ni (x,y) \to y \in \Omega_2$

are proper.

A properly supported FIO maps C_0^∞ to C_0^∞ and can be extended to a map from $\mathcal{D}'(\Omega_2)$ to $\mathcal{D}'(\Omega_1)$.

2.9. Pseudo Differential Operators.

A Pseudo Differential Operator (ΨDO) of order m is defined as a FIO with phase function $\langle x-y,\theta\rangle$ and symbol $s = s(x,y,\theta)$ in $S^m_{\rho,\delta}(\Omega \times \Omega \times \mathbb{R}^n)$. Here $n = \dim \Omega$. In particular, every PDO is a ΨDO (see example 2 in section 2.8).

Consider a ΨDO with kernel K given by

$$\int e^{i\langle x-y,\theta\rangle} s(x,y,\theta)d\theta, \quad s \in S^m_{\rho,\delta}(\Omega \times \Omega \times \mathbb{R}^n).$$

Then $WF(K) \subset \{(x,x,\theta,-\theta) \mid \theta \neq 0\}$.

Let $\chi(x,y)$ be a smooth function in $\Omega \times \Omega$, equal to 1 in a neighbourhood of the diagonal in $\Omega \times \Omega$ and properly supported. Then

$$K = \int e^{i\langle x-y,\theta\rangle}\chi(x,y)s(x,y,\theta)d\theta + \int e^{i\langle x-y,\theta\rangle}(1-\chi(x,y))s(x,y,\theta)d\theta.$$

Here the first integral on the right is properly supported and the other one is smoothing, that is, it maps $\mathcal{E}'(\Omega)$ to $C^\infty(\Omega)$. This follows from property (2.8.5).

Properly supported ΨDOs have another "standard form". If A is such an operator and $\delta < \rho$ then A can be written uniquely in the form

$$(2.9.1) \qquad Au(x) = \frac{1}{(2\pi)^n} \int e^{i\langle x,\eta\rangle}\sigma(x,\eta)\hat{u}(\eta)d\eta, \; u \in \mathcal{S}(\mathbb{R}^n), \; x \in \Omega.$$

Here $\sigma \in S^m_{\rho,\delta}(\Omega \times \mathbb{R}^n)$ is called the complete symbol of A. Asymptotically it is given by

$$(2.9.2) \qquad \sigma(x,\eta) \sim (2\pi)^n \sum_\alpha (iD_\eta)^\alpha (D_y)^\alpha s(x,y,\eta)/\alpha!\Big|_{y=x}.$$

For example, the complete symbol of a PDO is again $p(x,\theta)$.
For a properly supported ΨDO of order m with complete symbol σ in $S^m_{1,0}$, a principal symbol is defined as a symbol $\tau \in S^m_{1,0}$ so that $\sigma-\tau \in S^{m-1}_{1,0}$. For example, if σ is given by formula (2.9.2), then $(2\pi)^n s(x,x,\eta)$ is a principal symbol. Note that is not unique.

For a ΨDO A with kernel K WF(K) can be identified with

$$\{(x,\theta) \mid (x,x,\theta,-\theta) \in WF(K)\} =: WF(A).$$

Then the complete symbol σ contains the following information:
The complement of $WF(A)$ is the largest open cone in $T^*(\Omega) \setminus 0$ in which σ is rapidly decreasing.
This leads to: if A is as above and $u \in \mathcal{D}'(\Omega)$ then $WF(A) \cap WF(u) = \emptyset$ implies $Au \in C^\infty$.

* REMARK. Consider again convergence in $\mathcal{D}'_\Gamma(\Omega)$ (see section 2.6). Choose (φ,V) so that $\Gamma \cap (\text{supp}\,\varphi \times V) = \emptyset$. Let $\sigma = \sigma(\xi)$ be a symbol so that $\text{supp}(\sigma) \subset V$. Then A defined by

$$Au := \int e^{i<x,\xi>}\sigma(\xi)\widehat{\varphi u}(\xi)d\xi$$

is a (locally finite sum of properly supported) ΨDO(s) and $WF(A) \cap \Gamma = \emptyset$. So $Au \in C^\infty$ for $u \in \mathcal{D}'_\Gamma(\Omega)$ and the condition

$$\sup_V |\xi|^N|\widehat{\varphi u} - \widehat{\varphi u_j}| \to 0 \text{ for all N implies } Au_j \to Au \text{ in } C^\infty.$$

Moreover, if $Au_j \to Au$ in C^∞ for any such A, the converse also holds.

2.10. The bicharacteristic relation.

Let P be a properly supported ΨDO with a principal symbol $p \in S^m_{1,0}$ homogeneous of degree m for $\xi \neq 0$. For example, if $P = \Sigma_{|\alpha|\le m}\, a_\alpha(x)\frac{\partial^\alpha}{\partial x^\alpha}$, then $p(x,\xi) = \Sigma_{|\alpha|=m}\, a_\alpha(x)(i\xi)^\alpha$.

In global theories it is convenient to consider p as defined on $T^*(\Omega)$. In our situation $T^*(\Omega)$ can be identified with $\Omega \times \mathbb{R}^n$.

(2.10.1) $N := \{(x,\xi) \mid p(x,\xi) = 0,\ x \in \Omega,\ \xi \neq 0\}$ is called the characteristic set of P. It is a closed, conic subset of $T^*(\Omega) \setminus 0$. A surface in Ω defined by $\varphi(x) = c$ is called characteristic (at x) if $p(x,\varphi_x(x)) = 0$ when $\varphi(x) = c$ (at x). That is, the normal bundle (at x) is in N.
From now on we assume that p is real (modulo a constant).
Let

$$H_p := \sum_{k=1}^n \left(\frac{\partial p}{\partial \xi_k}\frac{\partial}{\partial x_k} - \frac{\partial p}{\partial x_k}\frac{\partial}{\partial \xi_k}\right).$$

H_p is called the Hamilton vector field of p.
If for all $(x,\xi) \in N$, $\Sigma\,(\partial p/\partial\xi_k)^2 + \Sigma\,(\partial p/\partial x_k)^2 \neq 0$, then N is a smooth

(2n-1)-dimensional submanifold of $\Omega \times (\mathbb{R}^n \setminus 0)$ and H_p is tangent to N. So it defines a flow on N, called the Hamilton flow of p. The integral curves of this flow are called the bicharacteristic strips of p. They are the solutions of the Hamilton-Jacobi equations

$$\frac{dx_j}{ds} = \frac{\partial p}{\partial \xi_j}, \quad \frac{d\xi_j}{ds} = -\frac{\partial p}{\partial x_j}, \quad j = 1,\dots,n, \quad p(x(s),\xi(s)) = 0.$$

If for some j, $\partial p/\partial \xi_j \neq 0$, the projections of these strips to Ω are smooth curves in Ω, called the bicharacteristic curves of p. A strip which has a given curve as projection will be referred to as a strip above that curve.

P is said to be of (real) principal type in Ω if P has a real homogeneous principal symbol and no complete strip stays over a compact set in Ω.

Ω is said to be pseudo- (or bicharacteristically) convex with respect to P if for every compact set $K \subset \Omega$ there is another compact set $K' \subset \Omega$ so that K' contains any interval on a bicharacteristic curve of P with both end points in K.

Finally, the bicharacteristic relation C is defined by

$$C := \{((x,\xi),(y,\eta)) \in N \times N \mid (x,\xi) \text{ and } (y,\eta) \text{ are on the same bicharacteristic strip}\}.$$

If P is of real principal type in Ω and Ω is pseudo convex with respect to P, then C is a closed conic submanifold of $(T^*(\Omega) \setminus 0) \times (T^*(\Omega) \setminus 0)$ which is closed in $T^*(\Omega \times \Omega) \setminus 0$.

2.11. Parametrices.

Let $\Omega_1 \subset \mathbb{R}^{n_1}$, $\Omega_2 \subset \mathbb{R}^{n_2}$ be open subsets and let $A : \mathcal{D}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$ be a properly supported continuous linear operator. If $B : \mathcal{E}'(\Omega_1) \to \mathcal{D}'(\Omega_2)$ is a continuous linear operator, then B is called a left parametrix of A if $BAu - u \in C^\infty(\Omega_2)$ for all $u \in \mathcal{E}'(\Omega_2)$. So $BA = I + R$ and R is a smoothing operator, that is, R has a smooth kernel (see section 2.7, case 1).
A right parametrix of A is a continuous linear operator $C : \mathcal{E}'(\Omega_1) \to \mathcal{D}'(\Omega_2)$ so that $ACu - u \in C^\infty(\Omega_1)$ for $u \in \mathcal{E}'(\Omega_1)$. So $AC = I + R_1$ and R_1 is smoothing.
A parametrix of A is a continuous linear operator which is both a left and a right parametrix of A.

If B is a left parametric of A and Au = f, $u \in \mathcal{E}'$, then Bf = BAu = u + Ru. So the possibility of singularities in u is completely determined by the singularities of B. See section 2.7, formula (2.7.6). If B is a right parametrix then for $f \in \mathcal{E}'$, A(Bf) = ABf = $f + R_1 f$, so modulo a smooth function, Bf is a solution of the equation Au = f. In case both A and B are properly supported these results extend to $u \in \mathcal{D}'$.

Ellipticity.

Let $\Omega \subset \mathbb{R}^n$, $s \in S^m_{\rho,\delta}(\Omega \times \mathbb{R}^N)$ and let Γ be an open conic subset of $\Omega \times (\mathbb{R}^N \setminus 0)$. $s = s(x,\theta)$ is said to be elliptic in Γ of order m if for every compact $K \subset \Omega$ there are constants C and R so that,

(2.11.1) $$|s(x,\theta)| \geq C|\theta|^m \text{ if } x \in K,\ |\theta| \geq R,\ (x,\theta) \in \Gamma.$$

If $\Gamma = \Omega \times (\mathbb{R}^N \setminus 0)$ we say s is elliptic.

A ΨDO A is called elliptic if it is defined by an elliptic symbol $s(x,y,\theta)$. Note that if this symbol is elliptic in Γ then the complete symbol σ is elliptic in $\{(x,\theta) \mid (x,x,\theta) \in \Gamma\}$ which is an open conic subset of $\Omega \times (\mathbb{R}^N \setminus 0)$. This can easily be seen by observing that it is sufficient that a principal symbol has property (2.11.1). So if $\rho > \delta$ every elliptic ΨDO is modulo a smoothing operator equal to a properly supported ΨDO A defined by equation (2.9.1) with an elliptic complete symbol σ. Such operators A have a parametrix, which can be constructed by successive approximations:

If $\chi(x,\theta)$ is zero in a neighbourhood of the zeros of σ and equal to one for $|\theta|$ large, the estimate (2.11.1) shows that $\frac{\chi}{\sigma}$ is an element of $S^{-m}_{\rho,\sigma}$.
Let Q be an ΨDO with symbol $\frac{\chi}{\sigma}$. Then it can be shown that QA − I = R is a ΨDO with symbol in $S^{-\rho+\delta}_{\rho,\delta}$. Since $-\rho+\delta < 0$, $E \sim I - R + R^2 - \dots$ for some ΨDO E with symbol in $S^0_{\rho,\delta}$. Then EQ is a left parametrix. In a similar way a right parametrix can be constructed. Both can be shown to be a parametrix for A.

If A is a properly supported ΨDO with complete symbol which is elliptic in an open conic subset containing (x_0,ξ_0), a similar construction shows that there is a ΨDO B so that $(x_0,\xi_0) \notin WF(BA - I)$ and $(x_0,\xi_0) \notin WF(AB - I)$. Such an operator is called a (microlocal) parametrix of A at (x_0,ξ_0).

A consequence of this fact is the following result. Let P be a PDO on Ω and N as in section 2.10, $u \in \mathcal{D}'(\Omega)$. Suppose $(x,\xi) \notin WF(Pu) \cup N$. Then there is a properly supported ΨDO Q so that $(x,\xi) \notin WF(QPu - u)$. Since

$u = QPu + (I - QP)u$ we have $WF(u) \subset WF(QPu) \cup WF(QPu - u)$. So $(x,\xi) \notin WF(u)$. Together with property 2 in section 2.4 this gives:

$$(2.11.2) \qquad WF(Pu) \subset WF(u) \subset WF(Pu) \cup N.$$

In particular, if P is elliptic: $WF(u) = WF(Pu)$.

Let now A be a properly supported FIO defined by a non-degenerate phase function φ and symbol $s = s(x,y,\theta) \in S^m_{\rho,1-\rho}(\Omega_1 \times \Omega_2 \times \mathbb{R}^N)$, $\rho > \frac{1}{2}$. Let $(x_0,\xi_0,y_0,\eta_0) \in \Lambda_\varphi$ (actually $(x_0,y_0,\xi_0,\eta_0) \in \Lambda_\varphi$: cf. section 2.7). A is said to be elliptic in (x_0,ξ_0,y_0,η_0) if for some open conic subsets $\Gamma \subset \Omega_1 \times \Omega_2 \times \mathbb{R}^N$, $(x_0,\xi_0) \in \Gamma_x \subset T^*(\Omega_1) \setminus 0$, $(y_0,\eta_0) \in \Gamma_y \subset T^*(\Omega_2) \setminus 0$ the map

$$C_\varphi \cap \Gamma \ni (x,y,\theta) \to (x,\varphi_x,y,\varphi_y) \in \Lambda_\varphi \cap (\Gamma_x \times \Gamma_y)$$

defines a diffeomorfism (cf. section 2.8) and s is elliptic in Γ. A is said to be elliptic if s is elliptic in $\Omega_1 \times \Omega_2 \times \mathbb{R}^N$. If A is an elliptic FIO, then A has a parametrix B which is a properly supported elliptic FIO, in general with different phase function. AB and BA are ΨDOs. If A is elliptic in (x_0,ξ_0,y_0,η_0) there is a properly supported FIO B, elliptic in (y_0,η_0,x_0,ξ_0) so that AB and BA are ΨDOs and $(x_0,\xi_0) \notin WF(AB - I_{\Omega_1})$, $(y_0,\eta_0) \notin WF(BA - I_{\Omega_2})$.

2.12. PDOs of real principal type.

In this section we discuss some results about the existence of solutions of PDEs of real principal type and their qualitative properties. These results are taken from Duistermaat/Hörmander [7], section 6.

We start with a general result on the propagation of singularities.

(2.12.1) Let P be a ΨDO on Ω, properly supported with real principal part p which is homogeneous of degree m. If $u \in \mathcal{D}'(\Omega)$ and $Pu = f$, then $WF(u) \setminus WF(f)$ is contained in N and is invariant under the Hamiltonian vector field H_p (see section 2.10).

The first statement already was given in section 2.11. The second means that if $(x_0,\xi_0) \in WF(u) \setminus WF(f)$, then the strip through (x_0,ξ_0) must be in $WF(u)$ at least until it hits $WF(f)$.

Let us sketch here a part of the proof. This proof is based on

reduction to the special case $P = D_n$ ($= \partial/i\partial x_n$).
Let $E_n^+ = i\delta(x'-y')H(x_n-y_n)$, $E_n^- = -i\delta(x'-y')H(y_n-x_n)$ denote the (kernels of) the forward and backward fundamental solutions of D_n. Then

$$(E_n^+ - E_n^-)u = $$
$$= (2\pi)^{-(n-1)} \int e^{i<x'-y',\theta>} iu(x,y)dxdyd\theta,\ u \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n).$$

For $\varphi = <x'-y',\theta>$ we have $\Lambda_\varphi = \{(x',x_n,\xi',0,x',y_n,-\xi',0) \mid \xi' \neq 0\}$. (For (x',ξ') fixed, the set $\{(x',x_n,\xi',0) \mid x_n \in \mathbb{R}\}$ describes a bicharacteristic strip of D_n!) It can be shown that this leads to:

$$WF'(E_n^\pm) = \Delta^* \cup \{(x',x_n,\xi',0,x',y_n,\xi',0) \mid \xi' \neq 0,\ x_n \gtrless y_n\}.$$

Here Δ^* = the diagonal in $(T^*(\mathbb{R}^n)\setminus 0) \times (T^*(\mathbb{R}^n)\setminus 0)$. Note that $(x,\xi,y,\eta) \in WF(K) \Leftrightarrow (x,\xi,y,-\eta) \in WF'(K)$ for any kernel K.

This result makes it possible to prove statement (2.12.1) for $P = D_n$ and $\Omega = \mathbb{R}^n$.

In the general case one can assume P to be of order one, for if Q is an elliptic ΨDO with positive principal part, homogeneous of degree 1-m, then QP has the same characteristics and bicharacteristic strips as P and WF(f) = WF(Qf) = WF(QPu).

If $(x_0,\xi_0) \in N$ and H_p has the direction of the cone axis $\{(x_0,\lambda\xi_0) \mid \lambda > 0\}$ in (x_0,ξ_0), then the strip through (x_0,ξ_0) is equal to this cone axis. So in that case, the statement is trivial. Therefore one can assume that H_p and this direction are linearly independent. Then for some $(y_0,\eta_0) \in T^*(\mathbb{R}^n)\setminus 0$, a conic neighbourhood Γ_x of (x_0,ξ_0) in $T^*(\Omega)\setminus 0$, a conic neighbourhood Γ_y of (y_0,η_0) in $T^*(\mathbb{R}^n)\setminus 0$ and a FIO A can be given with non-degenerate phase function φ, so that $(x_0,\xi_0,y_0,\eta_0) \in \Lambda'_\varphi$, A is elliptic in (x_0,ξ_0,y_0,η_0), the relation Λ'_φ maps the strip through (y_0,η_0) of D_n in Γ_y onto the strip of P through (x_0,ξ_0) in Γ_x and A transforms P locally to D_n in the sense that

$$(x_0,\xi_0,y_0,\eta_0) \notin WF'(PA - AD_n).$$

The operator A links the wave front sets of Pu and $D_n v$ and of u and v, where v is a distribution so that $u \equiv Av$ which exists because of the ellipticity of A. Then the general case can be deduced from the case $P = D_n$.

On the other hand there is the following statement:

(2.12.2) Let P be as in statement (2.12.1). Let $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to T^*(\Omega) \setminus 0$ be a map defining a strip which has at most one point in common with every cone axis. Let $\Gamma :=$ closure of $\{(x,\lambda\xi) \mid (x,\xi) \in \gamma(I), \lambda > 0\}$ in $T^*(\Omega) \setminus 0$, the closed conic hull of $\gamma(I)$.
Let $\Gamma' :=$ the limit points of $\Gamma := \cap_{I_0}$ closed conic hull of $\gamma(I \setminus I_0)$, with $I_0 \subset I$, I_0 compact. Then one can find $u \in \mathcal{D}'(\Omega)$ so that

$$WF(Pu) \subset \Gamma' \quad \text{and} \quad WF(u) \setminus \Gamma' = \Gamma \setminus \Gamma'.$$

If γ composed with the projection to Ω is proper, then $\Gamma' = \emptyset$ so

$$Pu \in C^\infty \quad \text{and} \quad WF(u) = \Gamma.$$

We now turn to some results on existence of solutions. Let P be as in statement (2.12.1).

(2.12.3) Let $K \subset \Omega$ be compact so that no complete bicharacteristic curve is contained in K. Then

$$\mathcal{N}(K) := \{v \in \mathcal{E}'(K) \mid {}^tPv = 0\}$$

is a finite dimensional subspace of $C_0^\infty(K)$ orthogonal to $P\mathcal{D}'(\Omega)$. If $f \in C^\infty(\Omega)$ and f is orthogonal to $\mathcal{N}(K)$ then one can find $u \in C^\infty(\Omega)$ so that $Pu = f$ in a neighbourhood of K.

That $\mathcal{N}(K) \subset C^\infty(\Omega)$ follows directly from statement (2.12.1) and the condition on the bicharacteristic curves.

The condition on the bicharacteristics is merely sufficient, but not necessary for the conclusions to be valid. For instance, if the condition is violated the lower order terms might rescue the conclusion. However, these terms are irrelevant when the condition is fulfilled. In particular, if P is of real principal type (see section 2.10), this condition is valid for every compact $K \subset \Omega$. In that case the following global solvability theorem can be proved:
If P is of real principal type in Ω then

P defines a surjective map from $\mathcal{D}'(\Omega)$ to $\mathcal{D}'(\Omega)/C^\infty(\Omega)$

$\Leftrightarrow \Omega$ is pseudo convex (see section 2.10) with respect to P.

This leaves open only the problem to solve Pu = f for $f \in C^\infty(\Omega)$. The following statement then holds:

The equation Pu = f has a solution $u \in C^\infty(\Omega)$ for all $f \in C^\infty(\Omega)$ so that $\langle f,v\rangle = 0$ for all $v \in C_0^\infty(\Omega)$ with ${}^tPv = 0$

$\Leftrightarrow$ For every compact $K \subset \Omega$ there is a compact $K' \subset \Omega$ so that $v \in E'(\Omega)$ and supp ${}^tPv \subset K$ implies ${}^tPv = {}^tPw$ for some $w \in E'(\Omega)$ with supp $w \subset K'$.

Finally we discuss the construction of parametrices for P.

The example of the case $P = D_n$ and the result (2.12.1) show that the bicharacteristic relation C as defined in section 2.10 will play an important role. Note that for $P = D_n$ the set Λ_φ is equal to the set C'! ($C' = \{(x,\xi,y,\eta) \mid (x,\xi,y,-\eta) \in C\}$.)

If P is of real principal type and Ω pseudo convex, then C is a closed conic subset of $T^*(\Omega \times \Omega) \setminus 0$, so it might serve as the wave front set of some kernel $K \in \mathcal{D}'(\Omega \times \Omega)$. Moreover, if P is of real principal type then:

Ω is pseudoconvex $\Leftrightarrow$
C is a closed conic submanifold of $T^*(\Omega \times \Omega) \setminus 0$ which is contained in $(T^*(\Omega) \setminus 0) \times (T^*(\Omega) \setminus 0)$ and $\forall\ (x_0,\xi_0,y_0,\eta_0) \in C$ there is a conic neighbourhood Γ so that $C \cap \Gamma = \Lambda'_\varphi$ for some non-degenerate phase function φ.

So locally C can be described by means of phase functions. This indicates the role FIOs will play in the construction of parametrices: If A is the kernel of a FIO with phase function φ defining C, then $WF(A) \subset \Lambda_\varphi \subset C'$!

Essentially the construction of a parametrix amounts to the following procedure:

Again we may assume that the symbol of P is of order 1, for an elliptic ΨDO has a parametrix which is a ΨDO. Note that for the identity operator I, $WF(I) = T^*(\Omega) \setminus 0$, that is, its kernel has $WF' = \Delta^*$, the diagonal in $(T^*(\Omega) \setminus 0) \times (T^*(\Omega) \setminus 0)$. Write I as sum of ΨDOs T_i with symbols in $S^0_{1,0}$ so that the supports of the corresponding kernels are locally finite. Moreover, $WF(T_i)$ should be in such a small open conic subset Γ_i of $T^*(\Omega) \setminus 0$ that either T_i is elliptic in Γ_i, in which case there is a ΨDO F_i with $PF_i = T_i + R_i$ with $R_i \in C^\infty$ (see section 2.11), or P can locally be transformed to D_n using locally elliptic FIOs (see the discussion of statement

(2.12.1)). In the second case, transformation of one of the fundamental solutions $E_n^{\pm}$ of D_n in the opposite direction gives an operator F_i so that $PF_i = T_i + R_i$. The supports of the F_i can be chosen locally finite so that $F = \Sigma F_i$ is welldefined and $PF = I + R$. Here R plays the same role as in the elliptic case, that is, for some G, $PG - R \in C^{\infty}$. But then $F - G$ is a right parametrix for P. A left parametrix for P can be obtained by noting that ${}^t({}^tP\,{}^tE) = EP$ and ${}^tP$ has the same principal symbol as P.

Note that if one chooses E_n^+ then E propagates singularities in Γ_i only in forward direction, if one chooses E_n^-, E propagates singularities in backward direction. In different Γ_i, different choices might be made. The number of different parametrices thus obtained depends on the number of components of $C \setminus \Delta_N$, Δ_N the diagonal in $N \times N$. In particular, define C^+ and C^- as the set of points $((x,\xi),(y,\eta))$ in $N \times N$ so that (x,ξ) lies after (resp. before) (y,η) on a bicharacteristic strip. Then $C \setminus \Delta_N = C^+ \cup C^-$, C^+, C^- are unions of components. Then the procedure given above leads to parametrices E^+ and E^- so that

$$WF'(E^+) = \Delta^* \cup C^+, \quad WF'(E^-) = \Delta^* \cup C^-.$$

Any left or right parametrix E with $WF'(E)$ contained in $\Delta^* \cup C^+$ or $\Delta^* \cup C^-$ is equal to E^+ or E^- modulo C^{∞}.
$E^+ - E^-$ is a locally finite sum of FIOs with phase functions φ_j so that $C' = \cup \Lambda_{\varphi_j}$ and symbols s_j in $S_{1,0}$ which have order $\frac{1}{2} - m + \frac{1}{2}(n - N_j)$, where N_j is the number of phase variables θ in $s_j(x,y,\theta)$.
For other unions of components, similar statements are valid.

We conclude this section with two more statements concerning parametrices.

1. Let E be a left or right parametrix for P. Then $\Delta^* \subset WF'(E)$. (Note that $WF(Pu) \subset WF(u)$!)
2. Let P have real homogeneous principal part. If $A \in \mathcal{D}'(\Omega \times \Omega)$, A is the associated operator and PA has smooth kernel, then $(x,\xi,y,\eta) \in WF'(A)$ and $\xi \neq 0$ implies $p(x,\xi) = 0$ and $B(x,\xi) \times \{(y,\eta)\} \subset WF'(A)$. Here $B(x,\xi)$ is the bicharacteristic strip through (x,ξ).

2.13. Homogeneous distributions on $\mathbb{R}$.

Define for $\alpha \in \mathbb{C}$ the function x_+^{α} by

$$x_+^\alpha := \begin{cases} x^\alpha & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Here $\log x$ is chosen to be real for $x > 0$. For $\mathrm{Re}\,\alpha > -1$ x_+^α is locally integrable so it defines a distribution on $\mathbb{R}$. Moreover, for $\varphi \in C_0^\infty(\mathbb{R})$, $\langle x_+^\alpha, \varphi\rangle$ is an analytic function in α for $\mathrm{Re}\,\alpha > -1$. Now, for $\mathrm{Re}\,\alpha > -1$

$$\langle x_+^\alpha,\varphi\rangle = \int_0^\infty x^\alpha \varphi(x)dx = \int_0^1 x^\alpha[\varphi(x) - \varphi(0)]dx + \int_1^\infty x^\alpha\varphi(x)dx + \frac{\varphi(0)}{\alpha+1}.$$

The expression on the right is welldefined for $\mathrm{Re}\,\alpha > -2$, $\alpha \neq -1$. In this way one obtains an analytic continuation of x_+^α for $\mathrm{Re}\,\alpha > -2$, $\alpha \neq -1$. Similarly one obtains an analytic continuation of x_+^α for $\mathrm{Re}\,\alpha > -n-1$, $\alpha \neq -1,-2,\dots,-n$. So x_+^α has an analytic continuation for $\alpha \in \mathbb{C}$, $\alpha \neq -1,-2,\dots$. This is denoted by x_+^α as well. $\langle x_+^\alpha,\varphi\rangle$ has poles of first order in $\alpha = -1,-2,\dots$. $(\alpha+n)x_+^\alpha \to \frac{(-1)^n}{(n-1)!}\,\delta^{(n-1)}_{(x=0)}$ for $\alpha \to -n$, $n = 1,2,\dots$.

Then x_+^α has the properties:

1. $x\cdot x_+^\alpha = x_+^{\alpha+1}$, $\alpha \neq -1,-2,-3,\dots$.
2. $\frac{d}{dx}x_+^\alpha = \alpha x_+^{\alpha-1}$, $\alpha \neq 0,-1,-2,-3,\dots$.
3. For $\lambda > 0$: $\langle x_+^\alpha,\varphi\rangle = \lambda^\alpha \langle x_+^\alpha, \varphi(x\lambda)\rangle$, $\alpha \neq -1,-2,\dots$.
   That is, x_+^α is homogeneous of degree α.

For $\mathrm{Re}\,\alpha > 0$, these properties are evident so they follow for other values of α by analytic continuation.

If $x_-^\alpha := \begin{cases} 0 & \text{if } x > 0 \\ |x|^\alpha & \text{if } x < 0 \end{cases}$,

then in a similar way x_-^α can be continuated analytically to a distribution x_-^α for $\alpha \neq -1,-2,\dots$. It has the same properties as x_+^α.

The distributions $(x+i0)^\alpha$ and $(x-i0)^\alpha$, $\alpha \neq -1,-2,\dots$, are given by:

$$(x\pm i0)^\alpha := x_+^\alpha + e^{\pm\pi i\alpha}x_-^\alpha.$$

Let $\log z := \log|z| + i\arg z$, $|\arg z| < \pi$. Then

$$(x+i0)^\alpha = \lim_{\varepsilon\downarrow 0}(x+i\varepsilon)^\alpha \quad\text{and}\quad (x-i0)^\alpha = \lim_{\varepsilon\downarrow 0}(x-i\varepsilon)^\alpha.$$

Here the convergence is in $\mathcal{D}'(\mathbb{R})$ and even in $\mathcal{S}'(\mathbb{P})$. Note that $(x \pm i\varepsilon)^\alpha$ are smooth functions. These limits exist when $\alpha = -1,-2,\dots$, too, and $(x \pm i0)^\alpha$ becomes entire in α. In particular:

$$\frac{1}{x \pm i0} = \mathrm{vp}\frac{1}{x} \mp i\pi\delta_{x=0}.$$

Finally, we denote by $x^{\alpha}_{[0,1]}$, $\alpha \neq -1,-2,-3,\dots$, the analytic continuation of the distribution given for Re $\alpha > -1$ by the function

$$x^{\alpha}_{[0,1]} = \begin{cases} x^{\alpha} & \text{if } 0 < x < 1 \\ 0 & \text{if } x < 0,\ x > 1 \end{cases}.$$

Note that $x^{\alpha}_{[0,1]}$ is not homogeneous and $\frac{d}{dx}x^{\alpha}_{[0,1]} = \alpha x^{\alpha-1}_{[0,1]} - \delta_{(x=1)}$.

<u>Fourier transforms.</u>

Homogeneous distributions belong to S'. For the Fourier transforms we have:

$$[x^{\alpha}_{+}]^{\wedge} = \Gamma(\alpha+1)e^{(\alpha+1)\pi i/2}(-\xi+i0)^{-\alpha-1} =$$

$$= \Gamma(\alpha+1)e^{-(\alpha+1)\pi i/2}(\xi-i0)^{-\alpha-1}.$$

$$[x^{\alpha}_{-}]^{\wedge} = \Gamma(\alpha+1)e^{(\alpha+1)\pi i/2}(\xi+i0)^{-\alpha-1}.$$

$$[(x+i0)^{\alpha}]^{\wedge} = \frac{2\pi}{\Gamma(-\alpha)}e^{\alpha\pi i/2}\xi_{+}^{-\alpha-1}, \quad \alpha \neq 0,1,2,\dots .$$

$$[(x-i0)^{\alpha}]^{\wedge} = \frac{2\pi}{\Gamma(-\alpha)}e^{-\alpha\pi i/2}\xi_{-}^{-\alpha-1}, \quad \alpha \neq 0,1,2,\dots .$$

<u>Wave front sets.</u>

$$WF(x^{\alpha}_{\pm}) = \{(0,\xi) \mid \xi \neq 0\}.$$

$$WF((x \pm i0)^{\alpha}) = \{(0,\xi) \mid \xi \gtrless 0\}, \quad \alpha \neq 0,1,2,\dots .$$

Note that the $(x \pm i0)^{\alpha}$ are distributions with the smallest possible (non-empty) wave front set.

<u>2.14. Additional notations and symbols.</u>

If we use the signs $\pm$, $\mp$, $\gtrless$ etc. in a formula this formula should be interpreted as two formulas compressed into one formula.

<u>Example</u>. $\pm a \mp b \gtrless c_{\pm}$ signifies $+a-b > c_{+}$ and $-a+b < c_{-}$.
So first read the signs above, then the signs below.

If we use the symbol C to denote a (fixed) constant, we allow ourselves to use the same symbol C for several different constants.

If $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, then $\Delta_x := \Sigma_{k=1}^{n} \partial^2/\partial x_k^2$.

Finally we mention some symbols also used in Watson [27]:

If $n \in \mathbb{N}$, $\alpha \in \mathbb{C}$ then:

1. $(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$.

   $(\alpha)_0 := 1$.

2. $(\alpha,n) := (-1)^n \dfrac{(\frac{1}{2}-\alpha)_n(\frac{1}{2}+\alpha)_n}{n!} \quad \left(= \dfrac{\Gamma(\alpha+n+\frac{1}{2})}{n!\,\Gamma(\alpha-n+\frac{1}{2})}\right)$.

   Here $\Gamma(z)$ is the Gamma function.

CHAPTER 3

THE TRICOMI OPERATOR ON $\mathbb{R}^{n+1}$

3.1. Introduction.

In this chapter we will construct a fundamental solution for the Tricomi operator $\mathcal{T}$.
For $u \in \mathcal{D}'(\mathbb{R}^{n+1})$ $\mathcal{T}u$ is given by

$$\mathcal{T}u = \left(\frac{\partial^2}{\partial t^2} + t \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}\right) u.$$

Coordinates in $\mathbb{R}^{n+1}$ are (x,t), $x \in \mathbb{R}^n$.
An operator $E : \mathcal{E}'(\mathbb{R}^{n+1}) \to \mathcal{D}'(\mathbb{R}^{n+1})$ is said to be a fundamental solution for $\mathcal{T}$ if E is continuous and

$$\mathcal{T}Eu = u = E\mathcal{T}u \quad \text{for all } u \in \mathcal{E}'(\mathbb{R}^{n+1}).$$

Note that a fundamental solution is a parametrix. However, it is exact. The symbol of $\mathcal{T}$ is given by $-\tau^2 - t|\xi|^2$. Here $|\xi| = (\Sigma_{k=1}^n \xi_k^2)^{\frac{1}{2}}$. The symbol is real and homogeneous of degree 2. Clearly $\mathcal{T}$ is elliptic for $t > 0$ and hyperbolic for $t < 0$.

The Tricomi operator is very wellknown. It is one of the most simple PDOs which are elliptic in one part of $\mathbb{R}^{n+1}$ and hyperbolic in another part. For $n = 1$ Green's functions and Riemann functions were constructed a long time ago (see Germain/Bader [11]), so that solutions for the equation $\mathcal{T}u = f$ could be given at least for sufficiently smooth f. Moreover, $\mathcal{T}$ is one of the most simple PDOs which are not either elliptic or hyperbolic but nevertheless of real principal type (see section 2.10 and section 3.2). Also $\mathbb{R}^{n+1}$ is pseudo convex with respect to $\mathcal{T}$. Therefore the theory as explained briefly in section 2.12 gives the existence of a number of parametrices.

The reason why we still want to discuss the construction of a fundamental solution is threefold.
In the first place we show that at least some of the parametrices referred to above can be given in concrete formulas. Moreover, the parametrices we obtain are even fundamental solutions so that we have exact solutions of the equation $Tu = f$ for arbitrary $f \in E'(\mathbb{R}^{n+1})$. Finally, the explicit formulas give us the opportunity to illustrate some notions and the construction given in section 2.12.

3.2. The bicharacteristic structure.

The bicharacteristic strips of T are given by the Hamilton-Jacobi equations:

$$\frac{dx_j}{ds} = -2t\xi_j, \quad \frac{d\xi_j}{ds} = 0, \quad \frac{dt}{ds} = -2\tau, \quad \frac{d\tau}{ds} = |\xi|^2, \quad j = 1,\dots,n,$$

under the condition $(-t|\xi|^2 - \tau^2)(s) = 0$.
For $(\xi,\tau) \neq (0,0)$ these equations form a non-degenerate system. The strip that starts for $s = 0$ in (x_0,t_0,ξ_0,τ_0), $t_0|\xi_0|^2 + \tau_0^2 = 0$, can be obtained as follows:

We have $\xi(s) \equiv \xi_0$ and if $\xi_0 = 0$ then $\tau_0 = 0$, so $\xi_0 \neq 0$. But then $t(s) = -|\xi_0|^{-2}\tau^2(s)$, $\tau(s) = |\xi_0|^2 s + \tau_0$ and one gets that the strip is given by

$$\left\{\left(x_0 + \frac{2}{3}\frac{\xi_0}{|\xi_0|}\left(|\xi_0|s + \frac{\tau_0}{|\xi_0|}\right)^3 - \frac{2}{3}\xi_0\frac{\tau_0^3}{|\xi_0|^4}, -\left(|\xi_0|s + \frac{\tau_0}{|\xi_0|}\right)^2, \xi_0, |\xi_0|^2 s + \tau_0\right) \mid s \in \mathbb{R}\right\}.$$

Looking at the t-coordinate only, it is easy to see that no complete strip stays over a compact set in $\mathbb{R}^{n+1}$, so T is of real principal type in $\mathbb{R}^{n+1}$. Moreover, $\mathbb{R}^{n+1}$ is pseudo convex with respect to T (see section 2.10).

If we use the τ-coordinate as parameter along the strip instead of s, it can also be described as

$$\left\{(x_0 + \frac{2}{3}\frac{\xi_0}{|\xi_0|^4}(\tau^3 - \tau_0^3), -\frac{\tau^2}{|\xi_0|^2}, \xi_0, \tau) \mid \tau \in \mathbb{R}\right\}.$$

From this we can derive that the bicharacteristic relation C of T is given by:

$$\left\{((x_0 + \frac{2}{3}\frac{\xi_0}{|\xi_0|^4}(\tau^3 - \tau_0^3), -\frac{\tau^2}{|\xi_0|^2}, \xi_0, \tau), (x_0,t_0,\xi_0,\tau_0)) \mid \xi_0 \neq 0, \; t_0|\xi_0|^2 + \tau_0^2 = 0\right\} =$$

$$= \left\{\left(\left(y + \frac{2}{3}\frac{\xi}{|\xi|^4}(\tau^3 - \sigma^3), -\frac{\tau^2}{|\xi|^2}, \xi, \tau\right), \left(y, -\frac{\sigma^2}{|\xi|^2}, \xi, \sigma\right)\right) \mid \xi \neq 0\right\}.$$

C is a closed conic C^∞-submanifold of $T^*(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \setminus 0$.
From this expression it is clear that

$$\varphi(x,t,y,s,\xi,\tau,\sigma) := \langle x-y,\xi\rangle + t\tau - s\sigma + \frac{1}{3}\frac{1}{|\xi|^2}(\tau^3 - \sigma^3), \quad \xi \neq 0,$$

is a non-degenerate phase function defining C. But from Lemma A.7.2 it is also clear that the following four functions define C for $t < 0$, $s < 0$, $\theta \neq 0$:

$$\varphi_{(\pm,+)}(x,y,t,s,\theta) := \langle x-y,\theta\rangle \pm \tfrac{2}{3}((-t)^{\frac{3}{2}} + (-s)^{\frac{3}{2}})|\theta|,$$
$$\varphi_{(\pm,-)}(x,y,t,s,\theta) := \langle x-y,\theta\rangle \pm \tfrac{2}{3}((-t)^{\frac{3}{2}} - (-s)^{\frac{3}{2}})|\theta|.$$

The bicharacteristic curves of T are the projections of the bicharacteristic strips to the (x,t)-space.
For $t_0 \le 0$ a curve through (x_0,t_0) is given by

$$\left\{\left(x_0 + \frac{2}{3}\frac{\xi}{|\xi|^4}(\tau^3 - \tau_0^3), -\frac{\tau^2}{|\xi|^2}\right) \mid \tau \in \mathbb{R},\ t_0 = -\frac{\tau_0^2}{|\xi|^2}\right\}, \quad \xi \neq 0.$$

For a point (x,t) on such a curve we have

$$\left|x - x_0 + \frac{2}{3}\frac{\xi}{|\xi|^4}\tau_0^3\right|^2 = \frac{4}{9}\frac{\tau^6}{|\xi|^6} = -\frac{4}{9}t^3.$$

Each curve has a cusp at $t = 0$. Note that every strip is smooth, even at $t = 0$. A curve lies in the plane through $(x_0,0)$ spanned by $(\xi,0)$ and $(0,1)$. Identifying this plane with $\mathbb{R}^2$, with $x - x_0 = \lambda\frac{\xi}{|\xi|}$ we get the figures:

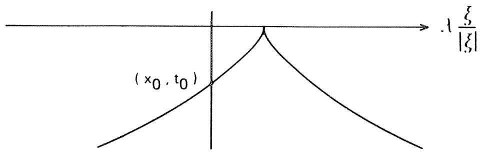

Fig. 1: bicharacteristic curve for $\tau_0 < 0$.

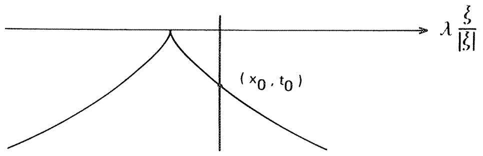

Fig. 2: bicharacteristic curve for $\tau_0 > 0$.

In figure 1 $\tau_0 < 0$, $\tau < 0$ on the left part of the curve and

$\tau > 0$ on the right part. In figure 2 $\tau_0 > 0$, $\tau > 0$ on the right part and $\tau < 0$ on the left part.

The set $\Lambda'_{\varphi_{(+,+)}}$ is given by

$$\Lambda'_{\varphi_{(+,+)}} = \left\{(y - \tfrac{2}{3}\frac{\theta}{|\theta|}((-t)^{\frac{3}{2}} + (-s)^{\frac{3}{2}}), t, \theta, -(-t)^{\frac{1}{2}}|\theta|; \; y, s, \theta, (-s)^{\frac{1}{2}}|\theta|) \mid \theta \neq 0,\ s < 0,\ t < 0\right\}.$$

For fixed (y,s,θ) this set describes the part of the strip through $(y,s,\theta,(-s)^{\frac{1}{2}}|\theta|)$ lying above the left part of the curve in figure 2. That is, the set $\{(y - \frac{2}{3}\frac{\theta}{|\theta|}((-t)^{\frac{3}{2}} + (-s)^{\frac{3}{2}}), t, \theta, -(-t)^{\frac{1}{2}}|\theta|) \mid t < 0\}$ is the indicated part of the strip.
Similarly the other phase functions are related to other parts of strips (and curves).

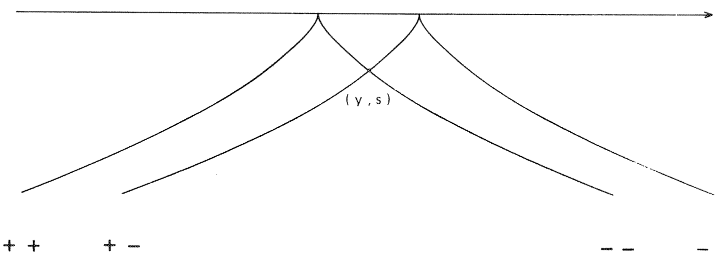

Fig. 3: relation between phase functions and curves.

3.3. The construction of a fundamental solution.

In this section we derive an explicit formula for a fundamental solution for T. We will proceed in two steps.
First we derive in a straightforward way a formal expression which hopefully has the properties we want. Second we show that indeed it has these properties.

So consider the equation $Tu = f$. We assume for this moment $f \in C_0^\infty(\mathbb{R}^{n+1})$. Partial Fourier transformation with respect to x gives the equation

$$\tilde{T}\tilde{u} = \tilde{f}. \tag{3.3.1}$$

Here $\tilde{T} = \partial^2/\partial t^2 - t|\xi|^2$, $\tilde{u} = \tilde{u}(\xi,t)$ and $\tilde{f} = \tilde{f}(\xi,t)$. Keeping ξ fixed, $\tilde{T}$ can be considered as an ordinary differential operator in t. The corresponding equation $\widetilde{T}\widetilde{u} = 0$ can be transformed by substituting $z = t|\xi|^{\frac{2}{3}}$ into the Airy equation

$$v'' - zv = 0, \qquad v = v(z).$$

Solutions of this equation are wellknown. See section A.2 about the Airy functions. Let $v_1(z)$ and $v_2(z)$ be two independent solutions of the Airy equation. Solutions of $\widetilde{T}\widetilde{u} = 0$ are given then by

$$c_1(\xi)v_1(t|\xi|^{\frac{2}{3}}) + c_2(\xi)v_2(t|\xi|^{\frac{2}{3}}).$$

Equation (3.3.1) can now be solved formally by the method of variation of constants. This gives

$$\widetilde{u}(\xi,t) = \frac{D}{|\xi|^{\frac{2}{3}}}\Big[v_1(t|\xi|^{\frac{2}{3}}) \int^{t} v_2(s|\xi|^{\frac{2}{3}})\widetilde{f}(\xi,s)ds - v_2(t|\xi|^{\frac{2}{3}}) \int^{t} v_1(s|\xi|^{\frac{2}{3}})\widetilde{f}(\xi,s)ds\Big].$$

Here $D = -\begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix}^{-1}$ is a constant.

The only solution of the Airy equation which is not exponentially increasing for $z \to +\infty$ is $Ai(z)$. It is even exponentially decreasing for $z \to +\infty$. Therefore, if we want to obtain u by means of inverse partial Fourier transformation, we must choose at any rate $v_1(z) = Ai(z)$ and reverse the interval of integration in the second integral. Thus we obtain the following tentative expression for a solution:

$$\text{(3.3.2)} \qquad u(x,t) = \frac{D}{(2\pi)^n} \int d\xi e^{i\langle x,\xi\rangle} \frac{1}{|\xi|^{\frac{2}{3}}} \times \Big[\int_{-\infty}^{t} Ai(t|\xi|^{\frac{2}{3}})v(s|\xi|^{\frac{2}{3}})\widetilde{f}(\xi,s)ds + \int_{t}^{\infty} v(t|\xi|^{\frac{2}{3}})Ai(s|\xi|^{\frac{2}{3}})\widetilde{f}(\xi,s)ds\Big].$$

Here $v = v(z)$ is a solution of the Airy equation not yet fixed.
We now shift our attention to the way this expression might propagate singularities in f in case $f \in \mathcal{E}'(\mathbb{R}^{n+1})$. In particular we are interested in values for $t \le 0$. The results of section A.5 show that $Ai(t|\xi|^{\frac{3}{2}})$ can be written as

$$Ai(t|\xi|^{\frac{2}{3}}) = e^{\pi i/3}a_-(t,\xi)e^{-\frac{2}{3}i(-t)^{\frac{3}{2}}|\xi|} + e^{-\pi i/3}a_+(t,\xi)e^{\frac{2}{3}i(-t)^{\frac{3}{2}}|\xi|}$$

for $t \neq 0$, $\xi \neq 0$. Here $a_\pm(t,\xi)$ are elliptic symbols for $t \neq 0$, $\xi \neq 0$. See Lemma A.5.1.

Also $v(t|\xi|^{\frac{2}{3}})$ can be expressed in such a way. The same exponentials appear. If we substitute these expressions in expression (3.3.2) we see that the phase functions $\varphi_{(\pm,+)}$ and/or $\varphi_{(\pm,-)}$ (see section 3.2) appear. So it might be possible to interpret expression (3.3.2) as a sum of FIOs multiplied by $H(t-s)$ or $H(s-t)$, at least for $s < 0$, $t < 0$. But then the phase functions determine the propagation of singularities. We will determine v so that a singularity in f is propagated at most along half a strip (cf. section 2.12, statement (2.12.1)). Looking at the pictures given in section 3.2, we note that we cannot allow all four phase functions to appear in the first integral in expression (3.3.2), for in that case they appear in the other integral as well and so $u(x,t)$ might become singular along an entire strip. This implies that we must choose

$$v(z) = Ai(e^{2\pi i/3}z) \quad \text{or} \quad v(z) = Ai(e^{-2\pi i/3}z) .$$

Since

$$Ai(e^{2\pi i/3}t|\xi|^{\frac{2}{3}}) = a_+(t,\xi)e^{\frac{2}{3}i(-t)^{\frac{3}{2}}|\xi|},$$

$$Ai(e^{-2\pi i/3}t|\xi|^{\frac{2}{3}}) = a_-(t,\xi)e^{-\frac{2}{3}i(-t)^{\frac{3}{2}}|\xi|}, \quad t < 0,\ \xi \neq 0,$$

substitution produces in the first integral the phase functions $\varphi_{(+,+)}$ and $\varphi_{(-,-)}$, in the second $\varphi_{(+,+)}$ and $\varphi_{(+,-)}$ or in the first integral the phase functions $\varphi_{(+,-)}$ and $\varphi_{(-,+)}$, in the second $\varphi_{(-,+)}$ and $\varphi_{(-,-)}$.

We will now define two operators which will be shown to have the properties we demand, that is, they are fundamental solutions and propagate singularities in a nice way:

$$(3.3.3) \qquad (A^\pm f)(x,t) := \frac{e^{\pm\pi i/6}}{(2\pi)^{n-1}} \int d\xi e^{i\langle x,\xi\rangle} \frac{1}{|\xi|^{\frac{2}{3}}} \times \left[\int_{-\infty}^{t} Ai(t|\xi|^{\frac{2}{3}})Ai(e^{\pm 2\pi i/3}s|\xi|^{\frac{2}{3}})\tilde{f}(\xi,s)ds + \int_{t}^{\infty} Ai(e^{\pm 2\pi i/3}t|\xi|^{\frac{2}{3}})Ai(s|\xi|^{\frac{2}{3}})\tilde{f}(\xi,s)ds \right].$$

Here $f \in C_0^\infty(\mathbb{R}^{n+1})$ and the constant D in formula (3.3.2) turned out to be $2\pi e^{\pm\pi i/6}$. Presumably a singularity of f in (y,s) can be propagated by $A^\pm$ only along that part of the strips above those parts of the curves given by the following pictures. We come back to this in the next section.

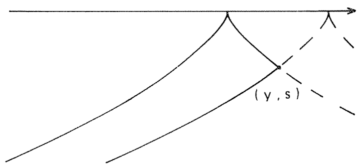

Fig. 4: propagation by A^+.

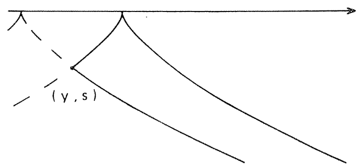

Fig. 5: propagation by A^-.

Let us now show that A^+ and A^- are fundamental solutions.

<u>LEMMA 3.3.4</u>. *Let* $f \in C_0^\infty(\mathbb{R}^{n+1})$ *and* $0 < T < \infty$.
$\forall\, n: \forall\, m: \exists\, C: \forall\, |t| \le T: \forall\, \xi:$

$$\left|\frac{\partial^n}{\partial t^n}\left(Ai(t|\xi|^{\frac{2}{3}}) \int_{-\infty}^{t} Ai(e^{\pm 2\pi i/3} s|\xi|^{\frac{2}{3}})\tilde{f}(\xi,s)ds\right)\right| \le C(1+|\xi|)^{-m},$$

$$\left|\frac{\partial^n}{\partial t^n}\left(Ai(e^{\pm 2\pi i/3} t|\xi|^{\frac{2}{3}}) \int_{t}^{\infty} Ai(s|\xi|^{\frac{2}{3}})\tilde{f}(\xi,s)ds\right)\right| \le C(1+|\xi|)^{-m}.$$

<u>PROOF</u>. It is clear that for every ξ fixed the expressions above are well-defined and smooth in t for $f \in C_0^\infty$.
Suppose $\mathrm{supp}(f) \subset \{(x,t) \mid x^2 + t^2 \le R^2\}$, $R > 0$. Then for some $C < \infty$

$$(3.3.5) \qquad |\xi|^{2m}\left|\int e^{-i\langle x,\xi\rangle} f(x,t)dx\right| = \left|\int_{|x|\le R} e^{-i\langle x,\xi\rangle}\Delta_x^m f(x,t)dx\right| \le C.$$

Here C is independent of t.
So $\tilde{f}(\xi,s)$ is rapidly decreasing in ξ uniformly in s.
Lemma A.5.2 shows that for $T_0 = \max\{R,T\}$

$$\forall\, n: \exists\, C_n: \left|\frac{\partial^n}{\partial t^n} Ai(e^{\pm 2\pi i/3}\, t|\xi|^{\frac{2}{3}})\right| \le \begin{cases} C_n(1+|\xi|)^{2n} e^{\frac{2}{3}|\xi| t^{\frac{3}{2}}}, & 0 \le t \le T_0 \\ C_n(1+|\xi|)^{2n}, & -T_0 \le t \le 0 \end{cases}.$$

$$\forall\, n: \exists\, C_n: \left|\frac{\partial^n}{\partial t^n} Ai(t|\xi|^{\frac{2}{3}})\right| \le \begin{cases} C_n(1+|\xi|)^{2n} e^{-\frac{2}{3}|\xi| t^{\frac{3}{2}}}, & 0 \le t \le T_0 \\ C_n(1+|\xi|)^{2n}, & -T_0 \le t \le 0 \end{cases}.$$

So

$$\left|\int_{-\infty}^{t} Ai(e^{\pm 2\pi i/3} s|\xi|^{\frac{2}{3}})\tilde{f}(\xi,s)ds\right| \le \int_{-\infty}^{t} |Ai(e^{\pm 2\pi i/3} s|\xi|^{\frac{2}{3}})|\, |\tilde{f}(\xi,s)|\, ds$$

$$\le \begin{cases} Ce^{\frac{2}{3}|\xi| t^{\frac{3}{2}}}(1+|\xi|)^{-m}, & 0 \le t \le T \\ C(1+|\xi|)^{-m}, & -T \le t \le 0 \end{cases}, \quad m \text{ arbitrary.}$$

Also

$$\left|\int_t^\infty Ai(s|\xi|^{\frac{2}{3}})\tilde{f}(\xi,s)ds\right| \le$$

$$\le \begin{cases} Ce^{-\frac{2}{3}|\xi|t^{\frac{3}{2}}}(1+|\xi|)^{-m}, & 0 \le t \le T \\ C(1+|\xi|)^{-m}, & -T \le t \le 0 \end{cases}, \quad m \text{ arbitrary.}$$

For $n = 0$ the statement now follows from the fact that the exponentials cancel. For $n > 0$ we use the estimates given above and an induction argument. □

COROLLARY 3.3.6. $A^{\pm}$ *maps* $C_0^\infty(\mathbb{R}^{n+1})$ *continuously to* $C^\infty(\mathbb{R}^{n+1})$.

PROOF. $\frac{\partial^\alpha}{\partial x^\alpha} e^{i<x,\xi>} = (i\xi)^\alpha e^{i<x,\xi>}$.
For fixed $\xi \neq 0$, the integrand is a smooth function in (x,t). For $|\xi| \ge 1$, Lemma 3.3.4 shows that every derivative of the integrand with respect to (x,t) can be bounded by an integrable function in ξ, uniformly in (x,t) for $|t| \le T$, $T < \infty$ arbitrary. Near $\xi = 0$ these derivatives behave not worse than $|\xi|^{-\frac{2}{3}}$ which is integrable. This implies $A^{\pm}f \in C^\infty$ for $f \in C_0^\infty$. The continuity follows from the fact that $f_j \to 0$ in C_0^∞ implies that the supports of the f_j are contained in a fixed compact set and

$$\left|\xi^\alpha \frac{\partial^n}{\partial t^n}\tilde{f}_j(\xi,t)\right| \le C^j_{\alpha,n} \to 0 \text{ for all } (\alpha,n).$$ □

By transposition ${}^tA^{\pm}$ is a continuous map from $\mathcal{E}'$ to $\mathcal{D}'$. Note that ${}^t\mathcal{T} = \mathcal{T}$.

LEMMA 3.3.7. ${}^tA^{\pm} = A^{\pm}$.

PROOF. For $f \in C_0^\infty$, $g \in C_0^\infty$: $<A^{\pm}f,g> = <f,A^{\pm}g>$. This follows easily after repeated application of Fubini's theorem using Lemma 3.3.4 and substitution $\xi \to -\xi$. □

PROPOSITION 3.3.8. $A^{\pm}$ extends to a fundamental solution for $\mathcal{T}$.

PROOF. For $u \in \mathcal{E}'$: $<A^{\pm}u,\varphi> := <u,A^{\pm}\varphi>$. It is clear that $A^{\pm}$ is welldefined for $u \in \mathcal{E}'$ and continuous. In order to show that $A^{\pm}$ is a fundamental solution it is sufficient to show that $A^{\pm}\mathcal{T}\varphi = \varphi = \mathcal{T}A^{\pm}\varphi$ for $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$. That $\mathcal{T}A^{\pm}\varphi = \varphi$ follows from a straightforward computation. $A^{\pm}\mathcal{T}\varphi = \varphi$ follows after repeated partial integration. □

3.4. Singularities of $A^{\pm}$.

In this section we discuss the way $A^{\pm}$ propagates singularities, as indicated in section 3.3. To this end we look for FIO-representations of $A^{\pm}$. Let $f \in \mathcal{E}'(\mathbb{R}^{n+1})$. Since $\mathcal{T}(A^{\pm}f) = f$, formula (2.11.2) shows that

$$\text{(3.4.1)} \qquad WF(f) \subset WF(A^{\pm}f) \subset WF(f) \cup N.$$

Moreover, $WF(A^{\pm}f) \setminus WF(f)$ is invariant under the Hamilton flow determined by $\mathcal{T}$. We intend to show that $A^{\pm}$ propagates a singularity of f in a point $(x,t,\xi,\tau) \in N$ only along the strip through that point and only in one direction. Also, if f is smooth along a strip, then so is $A^{\pm}f$.

For every $\varepsilon > 0$ f can be written as $f = f_- + f_0 + f_+$, $\text{supp}(f_{\pm}) \subset \{(x,t) \mid t \gtrless \pm\frac{\varepsilon}{2}\}$ and $\text{supp}(f_0) \subset \{(x,t) \mid |t| < \varepsilon\}$. Formula (3.4.1) holds for $f_{\pm}$ and f_0 as well. It is not difficult to check that $A^{\pm}f_+$ is smooth for $t < \frac{\varepsilon}{2}$. The set $N = \{(x,t,\xi,\tau) \mid t|\xi|^2 + \tau^2 = 0\}$ lies above $\{(x,t) \mid t \le 0\}$. Therefore we can restrict ourselves to the analysis of the singularities of $A^{\pm}f_-$ and $A^{\pm}f_0$ for $t \le 0$.

First we analyse $A^{\pm}f$ for $f \in \mathcal{E}'$ with $\text{supp}(f) \subset \{(x,t) \mid t < 0\}$. Choose $\chi = \chi(\xi) \in C_0^{\infty}(\mathbb{R}^n)$ so that $0 \le \chi(\xi) \le 1$ and

$$\chi(\xi) = \begin{cases} 1 & \text{for } |\xi| < 1 \\ 0 & \text{for } |\xi| > 2 \end{cases}.$$

Then $1-\chi$ is zero in a neighbourhood of $\xi = 0$. The functions χ and $1-\chi$ are called cut-off functions. In formula (3.3.3) we now replace the factor $|\xi|^{-\frac{2}{3}}$ by

$$\text{(3.4.2)} \qquad \frac{1}{|\xi|^{\frac{2}{3}}} = \chi(\xi)\frac{1}{|\xi|^{\frac{2}{3}}} + (1-\chi(\xi))\frac{1}{|\xi|^{\frac{2}{3}}}.$$

Then we can write $A^{\pm}f = A_1^{\pm}f + A_2^{\pm}f$, $A_1^{\pm}$ and $A_2^{\pm}$ are related to the first and second term in expression (3.4.2) respectively. Consider the operators $A^{\pm}$, $A_1^{\pm}$ and $A_2^{\pm}$ as operators from $\mathcal{E}'(\mathbb{R}^n \times \mathbb{R}^-)$ to $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^-)$.

$$f \to \int d\xi e^{i\langle x,\xi\rangle} \frac{\chi(\xi)}{|\xi|^{\frac{2}{3}}} \left[\int_{-\infty}^{t} Ai(t|\xi|^{\frac{2}{3}}) Ai(e^{2\pi i/3} s|\xi|^{\frac{2}{3}}) \tilde{f}(\xi,s) \right]$$

defines a continuous map between $C_0^{\infty}(\mathbb{R}^n \times \mathbb{R}^-)$ and $C^{\infty}(\mathbb{R}^n \times \mathbb{R}^-)$. It has the kernel $K = K(x,t,y,s)$ given by

$$H(t-s) \int d\xi e^{i\langle x-y,\xi\rangle} \frac{\chi(\xi)}{|\xi|^{\frac{2}{3}}} Ai(t|\xi|^{\frac{2}{3}}) Ai(e^{2\pi i/3} s|\xi|^{\frac{2}{3}}).$$

Because χ has compact support the integral defines a smooth function in (x,t,y,s). So the singularities of K are given by those of $H(t-s)$. $H(t-s)$ is the pullback of H under the map $(x,t,y,s) \to t-s$. So the results given in section 2.6 show that

$$\text{(3.4.3)} \qquad WF(H(t-s)) = \{(x,s,0,\sigma;y,s,0,-\sigma) \mid \sigma \neq 0\}.$$

A similar analysis of the other terms shows that the kernels of A_1^+ and A_1^- both have their wave front set in this set. Note that $(x,s,0,\sigma) \notin N$ for all (x,s,σ). Formula (2.7.6) in section 2.7 now shows that A_1^+ and A_1^- do not propagate singularities along a bicharacteristic strip.

Next we discuss for $s < 0$, $t < 0$ the kernel given by

$$\int d\xi e^{i\langle x-y,\xi\rangle} \frac{1-\chi(\xi)}{|\xi|^{\frac{2}{3}}} Ai(t|\xi|^{\frac{2}{3}}) Ai(e^{2\pi i/3} s|\xi|^{\frac{2}{3}}).$$

Here $(1-\chi)/|\xi|^{\frac{2}{3}}$ is zero in a neighbourhood of $\xi = 0$. The formulas given in the previous section show that this kernel can be written as the sum of two oscillatory integrals with phase functions $\langle x-y,\xi\rangle - \frac{2}{3}|\xi|((-t)^{\frac{3}{2}} - (-s)^{\frac{3}{2}})$, $\langle x-y,\xi\rangle + \frac{2}{3}|\xi|((-t)^{\frac{3}{2}} + (-s)^{\frac{3}{2}})$ and symbols

$$\frac{1-\chi(\xi)}{|\xi|^{\frac{2}{3}}} a_-(t,\xi)a_+(s,\xi), \qquad \frac{1-\chi(\xi)}{|\xi|^{\frac{2}{3}}} a_+(t,\xi)a_+(s,\xi)$$

respectively.
The symbols are elements of $S^{-1}_{1,0}$. Of course, these phase functions are the functions $\varphi_{(-,-)}$ and $\varphi_{(+,+)}$ from section 3.2. The kernel given for $s < 0$, $t < 0$ by

$$\int d\xi e^{i\langle x-y,\xi\rangle} \frac{1-\chi(\xi)}{|\xi|^{\frac{2}{3}}} Ai(e^{2\pi i/3} t|\xi|^{\frac{2}{3}}) Ai(s|\xi|^{\frac{2}{3}})$$

can be written as the sum of two oscillatory integrals with phase functions $\varphi_{(+,+)}$, $\varphi_{(+,-)}$ and symbols

$$\frac{1-\chi}{|\xi|^{\frac{2}{3}}} a_+(t,\xi)a_+(s,\xi), \quad \frac{1-\chi}{|\xi|^{\frac{2}{3}}} a_+(t,\xi)a_-(s,\xi)$$

respectively, which are again in $S^{-1}_{1,0}$.
But then it follows that the kernel of A_2^+ restricted to $\{(x,t,y,s) \mid s < 0, t < 0\}$ is a linear combination of the kernels

$$\int e^{i\varphi(+,+)} \frac{1-\chi}{|\xi|^{\frac{2}{3}}} a_+(t,\xi)a_+(s,\xi)d\xi,$$

$$H(t-s) \int e^{i\varphi(-,-)} \frac{1-\chi}{|\xi|^{\frac{2}{3}}} a_-(t,\xi)a_+(s,\xi)d\xi \quad \text{and}$$

$$H(s-t)\int e^{i\varphi(+,-)}\frac{1-\chi}{|\xi|^{\frac{2}{3}}}a_+(t,\xi)a_-(s,\xi)d\xi.$$

The results of section 2.6 on multiplication of distributions show that multiplication with $H(\pm(t-s))$ is welldefined. This follows from equation (3.4.3) and the fact that $\partial\varphi(\pm,\pm)/\partial x = \xi \neq 0$. The wave front sets of these kernels are contained in

$$\Lambda_{\varphi(+,+)},\ \Lambda_{\varphi(-,-)}\Big|_{t\geq s} \cup \mathrm{WF}\,H(t-s) \cup$$
$$\{(y,s,\xi,\sigma+(-s)^{\frac{1}{2}}|\xi|;\ y,s,-\xi,-\sigma-(-s)^{\frac{1}{2}}|\xi|)\ |\ \sigma\neq 0,\ \xi\neq 0,\ s<0\} \quad \text{and}$$

$$\Lambda_{\varphi(+,-)}\Big|_{t\leq s} \cup \mathrm{WF}\,H(s-t) \cup$$
$$\{(y,s,\xi,\sigma-(-s)^{\frac{1}{2}}|\xi|;\ y,s,-\xi,-\sigma+(-s)^{\frac{1}{2}}|\xi|)\ |\ \sigma\neq 0,\ \xi\neq 0,\ s<0\}$$

respectively.
It is clear that the only relevant sets are $\Lambda_{\varphi(+,+)}$, $\Lambda_{\varphi(-,-)}$ and $\Lambda_{\varphi(+,-)}$. The sets $\Lambda_{\varphi(+,+)} \cup \Lambda_{\varphi(-,-)}\big|_{t\geq s}$ and $\Lambda_{\varphi(+,-)}\big|_{t\leq s}$ show that a singularity in $(y,s,\xi,(-s)^{\frac{1}{2}}|\xi|)$ or $(y,s,\xi,-(-s)^{\frac{1}{2}}|\xi|)$ respectively, is propagated only along the strip through that point and only in one direction.

A similar result holds for the operator A_2^- (all directions are reversed).

The representation of $A^\pm$ given above is valid only for $s < 0$, $t < 0$ since the related phase functions describe the bicharacteristic relation only for $s < 0$, $t < 0$.

Let us now discuss $A^\pm f$ for f such that $\mathrm{supp}(f) \subset \{(x,t)\ |\ |t| < \varepsilon\}$. Again it is sufficient to determine the singularities of $A^\pm f$ for $t < 0$. For $\varphi \in C_0^\infty$ with support in $\mathbb{R}^n \times \mathbb{R}^-$ we write

$$A^\pm\varphi(x,t) = \frac{e^{\pm\pi i/6}}{(2\pi)^{n-1}}\int d\xi e^{i\langle x,\xi\rangle}\frac{1}{|\xi|^{\frac{2}{3}}} \times$$
$$\int_t^\infty \Big[-\mathrm{Ai}(t|\xi|^{\frac{2}{3}})\mathrm{Ai}(e^{\pm2\pi i/3}s|\xi|^{\frac{2}{3}}) + \mathrm{Ai}(e^{\pm2\pi i/3}t|\xi|^{\frac{2}{3}})\mathrm{Ai}(s|\xi|^{\frac{2}{3}})\Big]\tilde{\varphi}(\xi,s)ds$$
$$+ \frac{e^{\pm\pi i/6}}{(2\pi)^{n-1}}\int d\xi e^{i\langle x,\xi\rangle}\frac{1}{|\xi|^{\frac{2}{3}}}\int \mathrm{Ai}(t|\xi|^{\frac{2}{3}})\mathrm{Ai}(e^{\pm2\pi i/3}s|\xi|^{\frac{2}{3}})\tilde{\varphi}(\xi,s)ds\ .$$

Both integrals on the right are welldefined for all (x,t) and define smooth functions. They define continuous operators $A_3^\pm$ and $A_4^\pm$ respectively, from $C_0^\infty(\mathbb{R}^n \times \mathbb{R}^-)$ to $C^\infty(\mathbb{R}^{n+1})$. So for $f \in \mathcal{E}'$, $A^\pm f|_{t<0} = ({}^tA_3^\pm + {}^tA_4^\pm)f$.

First consider ${}^tA_3^\pm f$.
The kernel of ${}^tA_3^\pm$ has support in $\{(x,t,y,s) \mid s \leq t < 0\}$. Therefore we can as before use the asymptotic expansions of the Airy functions to show that the kernel of ${}^tA_3^+$ has its wave front set in

$$[\Lambda_{\varphi_{(-,-)}} \cup \Lambda_{\varphi_{(+,-)}}]\big|_{t\geq s} \cup \mathrm{WF}\, H(t-s) \cup$$
$$\{(y,t,\xi,\sigma;\ y,t,-\xi,-\sigma) \mid \xi \neq 0,\ t < 0\}.$$

So a singularity of f in (y,s,η,σ) causes ${}^tA_3^+ f|_{t<0}$ to be singular at most along that part of the strip through that point indicated in Fig. 6.

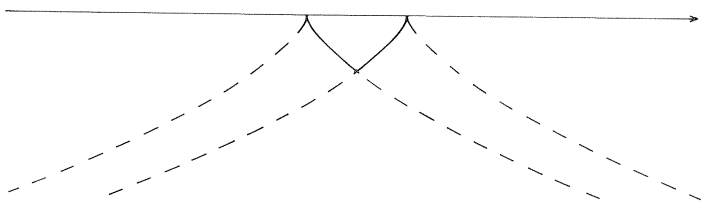

Fig. 6: propagation by ${}^tA_3^+$.

The same story can be told for ${}^tA_3^-$.

Next consider ${}^tA_4^+$.
Note $T({}^tA_4^+ f) = 0$ for $t < 0$, so $\mathrm{WF}({}^tA_4^+ f) \subset N|_{t<0}$ and is invariant under the Hamilton flow of T restricted to $t < 0$. As before it is sufficient to analyse the expression defining A_4^+ with $|\xi|^{-\frac{2}{3}}$ replaced by $(1-\chi(\xi))|\xi|^{-\frac{2}{3}}$. Using formula (A.2.1) it can be shown that for $\xi \neq 0$:

$$\text{(3.4.4)} \qquad \mathrm{Ai}(e^{2\pi i/3} s|\xi|^{\frac{2}{3}}) =$$
$$\frac{1}{2\pi|\xi|^{\frac{2}{3}}}\left[e^{\pi i/3}\int_{-\infty}^{0} e^{i(s\sigma+\frac{1}{3}\sigma^3/|\xi|^2)}\,d\sigma + e^{-\pi i/6}\int_{-\infty}^{0} e^{-s\sigma+\frac{1}{3}\sigma^3/|\xi|^2}\,d\sigma\right].$$

From this it follows that for $\psi(s) \in C_0^\infty(\mathbb{R}^-)$

$$\int e^{-is\sigma}\psi(s)\mathrm{Ai}(e^{2\pi i/3}s|\xi|^{\frac{2}{3}})ds$$

is rapidly decreasing in all directions (ξ_0,σ_0) with $\sigma_0 > 0$.
Then it follows easily that for $\varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^-)$

$$[\varphi({}^tA_4^+ f)]^\wedge$$

is rapidly decreasing in all directions (ξ_0,σ_0) with $\sigma_0 > 0$.

So $WF({}^tA_4^+f) \subset \{(x,t,\xi,\tau) \mid t < 0 \text{ and } \tau < 0\}$. That is, ${}^tA_4^+f$ can only be singular along the strips given in Fig. 7.

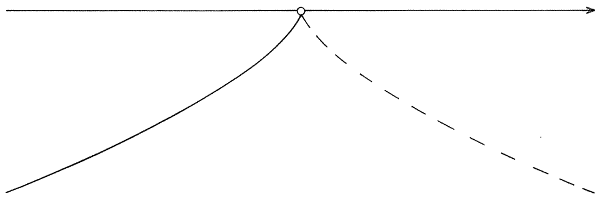

Fig. 7: propagation by ${}^tA_4^+$.

Now we return to A^+f, $\text{supp}(f) \subset \{(x,t) \mid |t| < \varepsilon\}$. We must show that if A^+f is singular in a point $(x_0,t_0,\xi_0,\tau_0) \in N \setminus WF(f)$, then f is singular in a point lying on the strip through that point and on the right side of (x_0,t_0,ξ_0,τ_0) in Fig. 8.

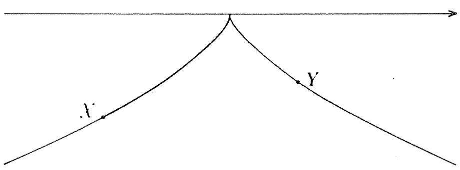

Fig. 8: A^+f singular in X then f singular in some point Y.

Well, if this would not be the case then A^+f is singular along an infinite part of the strip through (x_0,t_0,ξ_0,τ_0) on which the τ-coordinate is positive. This is impossible as follows from the analysis of ${}^tA_3^+$ and ${}^tA_4^+$. This concludes the analysis of A^+. A^- is treated in a similar way (all directions are reversed).

REMARK. In the analysis of ${}^tA_4^+$ we did not determine the singularities of the kernel of ${}^tA_4^+$ as we did for the kernel of ${}^tA_3^+$. This was not necessary. However, integral representations for the Airy functions like expression (3.4.4) lead after substitution at least locally to FIO-representations.

3.5. Connections between T and D_t.

T is an operator of real principal type, so T is locally equivalent to the operator D_t $(= \frac{\partial}{i\partial t})$. That is, for every characteristic point

$X = (x_0,t_0,\xi_0,\tau_0)$ of T a characteristic point Y of D_t can be found and a FIO A with properties as given in section 2.12, so that $(X,Y) \notin WF(P_0TA - AD_t)$. Here P_0 is a ΨDO with symbol of order (-1), elliptic in a conic neighbourhood of X. We will illustrate here in a sketchy way how these operators P_0 and A can be obtained, without going in too many technical details.

Note that the symbol of D_t is τ and that a strip of D_t through $(y_0,s_0,\eta_0,0)$ is given by $\{(y_0,s,\eta_0,0) \mid s \in \mathbb{R}\}$. Now let (x_0,t_0,ξ_0,τ_0) be a characteristic point of T. So $t_0|\xi_0|^2 + \tau_0^2 = 0$.

First we assume $t_0 = 0$. Then $\tau_0 = 0$ and $\xi_0 \neq 0$. The strip through $(x_0,0,\xi_0,0)$ is given by $\{(x_0 + \frac{2}{3}\tau^3\xi_0/|\xi_0|^4, -\tau^2/|\xi_0|^2, \xi_0, \tau) \mid \tau \in \mathbb{R}\}$. Let Γ be an open conic neighbourhood of $(\xi_0,0)$, so that $(\xi,\tau) \in \Gamma$ implies $\xi \neq 0$. Choose a cut-off function $\chi = \chi(\xi,\tau)$ so that $\chi = 1$ in Γ and $\chi = 0$ in a conic neighbourhood of $\xi = 0$. The symbol of T can be factorized as $|\xi|(-t|\xi| - \tau^2/|\xi|)$. Here $\chi(\xi,\tau)|\xi|$ is a symbol which is elliptic in Γ. Let P_0 be the ΨDO with symbol $\chi(\xi,\tau)\frac{1}{|\xi|}$ and let T_0 be the ΨDO with symbol $\chi(\xi,\tau)(-t|\xi| - \tau^2/|\xi|)$. Then it can easily be shown, using formulas for the composition of ΨDOs, that $P_0T - T_0$ is a ΨDO with wave front set disjunct from $\mathbb{R}^{n+1} \times \Gamma$. Therefore we can consider T_0 instead of T. Note that T_0 has the same bicharacteristic structure near $(x_0,0,\xi_0,0)$ as T has.

We will now give a phase function for A.
The phase functions occurring in section 3.2 invite us to consider the phase function

$$\psi(x,t,y,s,\xi,\tau,\sigma) = \langle x-y,\xi\rangle + t\tau - s\sigma - \frac{\tau\sigma}{|\xi|} + \frac{1}{3}\frac{\tau^3 - \sigma^3}{|\xi|^2}.$$

It is welldefined outside a conic neighbourhood of $\xi = 0$, non-degenerate and it meets the conditions on the critical points (cf. section 2.8).
We have

$$\frac{\partial\psi}{\partial\xi} = x - y + \frac{\xi}{|\xi|^3}\tau\sigma - \frac{2}{3}\frac{\xi}{|\xi|^4}(\tau^3 - \sigma^3),$$

$$\frac{\partial\psi}{\partial\tau} = t - \frac{\sigma}{|\xi|} + \frac{\tau^2}{|\xi|^2},$$

$$\frac{\partial\psi}{\partial\sigma} = -s - \frac{\tau}{|\xi|} - \frac{\sigma^2}{|\xi|^2}.$$

So

$$\Lambda'_\psi = \left\{\left(y + \tfrac{1}{3}\,\frac{\sigma^3\xi}{|\xi|^4} + \frac{s\sigma\xi}{|\xi|^2} - \tfrac{2}{3}\,\frac{\xi}{|\xi|}\left(s + \frac{\sigma^2}{|\xi|^2}\right)^3,\right.\right.$$
$$\left.\left.\frac{\sigma}{|\xi|} - \left(s + \frac{\sigma^2}{|\xi|^2}\right)^2, \xi, -s|\xi| - \frac{\sigma^2}{|\xi|};\ y, s, \xi, \sigma\right) \mid \xi \neq 0\right\}.$$

Substitution of $\sigma = 0$ gives

$$\{(y - \tfrac{2}{3}\,\frac{\xi}{|\xi|}\,s^3, -s^2, \xi, -s|\xi|;\ y, s, \xi, 0) \mid \xi \neq 0\}$$

and indeed, for fixed (y,ξ) on the right of this expression we see a strip of D_t and on the left a strip of T_0 (and T) (substitute $-s|\xi| = \tau$).
In order to get an elliptic FIO A such that

$$((x_0,0,\xi_0,0),(x_0,0,\xi_0,0)) \notin WF(T_0A - AD_t)$$

we choose ψ as the phase function and must determine a symbol. Because of the particular choice for the phase function ψ of A it is not difficult to show that we can take $a \equiv 1$, for instance. In order to verify this we need a formula for the composition of a ΨDO and a FIO given for instance in Trèves [25], page 332. We will not discuss this in further detail.

Next we assume $t_0 < 0$. Then $\tau_0 = \pm\sqrt{-t_0}|\xi_0|$.
We consider the case $\tau_0 = -\sqrt{-t_0}|\xi_0|$. Note that $-t|\xi|^2 - \tau^2 = (\sqrt{-t}|\xi| - \tau)(\sqrt{-t}|\xi| + \tau)$, $(x_0,t_0,\xi_0,-\sqrt{-t_0}|\xi_0|)$ is a characteristic point of $\sqrt{-t}|\xi| + \tau$ and $\sqrt{-t}|\xi| - \tau$ is an elliptic symbol in a conic neighbourhood of this point. Let P_1 be a ΨDO with symbol $\chi_1(t,\xi,\tau)(\sqrt{-t}|\xi| - \tau)^{-1}$, χ_1 an appropriate cut-off function. Then $P_1T = T_1 + R$, where R is a ΨDO with wave front set disjunct from a conic neighbourhood of $(x_0,t_0,\xi_0,-\sqrt{-t_0}|\xi_0|)$ and T_1 is a ΨDO with principal symbol $\chi_1(t,\xi,\tau)(\sqrt{-t}|\xi| + \tau)$. This is not the complete symbol since $\sqrt{-t}|\xi| - \tau$ depends on τ. Again we can consider T_1 instead of T.

Consider the phase function

$$\psi_1(x,t,y,s,\xi,\sigma) = \langle x-y,\xi\rangle + \tfrac{2}{3}|\xi|((-t)^{\frac{3}{2}} - (-t_0)^{\frac{3}{2}}) - (t-s)\sigma.$$

Then

$$\Lambda'_{\psi_1} =$$
$$\left\{(y - \tfrac{2}{3}\,\frac{\xi}{|\xi|}((-s)^{\frac{3}{2}} - (-t_0)^{\frac{3}{2}}), s, \xi, -\sqrt{-s}|\xi| - \sigma;\ y,s,\xi,-\sigma) \mid s < 0,\ \xi \neq 0\right\}.$$

For $\sigma = 0$ and (y,ξ) fixed, again we see the link between a strip of D_t and

a strip of $\mathcal{T}_1$ (and $\mathcal{T}$).
An elliptic FIO B with phase function ψ_1 can be constructed (locally) so that

$$((x_0,t_0,\xi_0,-\sqrt{-t_0}|\xi_0|),(x_0,t_0,\xi_0,0)) \notin WF(\mathcal{T}_1 B - BD_t).$$

As a principal symbol for B we can choose $b \equiv 1$.
A complete symbol can be obtained by means of a recursion procedure.

<u>REMARK</u>. The "time-variable" s is used as parameter along a strip of D_t and along a strip of $\mathcal{T}$ for $t < 0$. In the fundamental solution as constructed this is represented by the function $H(t-s)$. In a neighbourhood of $t = 0$, τ should be used as parameter (a strip reflects at $t = 0$!). Therefore one expects the function $H(\tau-\sigma)$ to appear in a FIO-representation of the fundamental solution near $t = 0$. Indeed, the analysis made in section 3.4 shows that $A^{\pm}$ propagates a singularity of f in (y,s,η,σ) only to points (x,t,ξ,τ) with $\tau \lesseqgtr \sigma$.

CHAPTER 4

BOUNDARY VALUE PROBLEMS FOR THE TRICOMI EQUATION

4.1. Introduction.

In this chapter we will study several boundary value problems for the Tricomi equation in $\mathbb{R}^2$.
Solutions for the problems discussed in sections 4.2 and 4.3 are wellknown in the case of sufficiently smooth boundary data. So is a solution for the problem discussed in section 4.5 with the boundary condition on $t = 1$ replaced by a condition on the behaviour of the solution for $x^2 + t^2 \to \infty$ $(t \geq 0)$. See for instance Bizadse [2] and Von Wolfersdorf [28].

We attempt to allow general distributional data. Moreover we wish to emphasize the use of Fourier techniques. This means that we will use Fourier transformation so that in a natural way FIOs will appear. The theory connected with these operators enables us to describe the qualitative properties of expressions and solutions. This in its turn can be used for instance to show that restrictions to the boundary are welldefined.

We also considered the possibility of extending the results thus obtained to regions with more general boundaries. Our findings for $t < 0$ (the hyperbolic part) are embodied in section 4.4. We only find solutions modulo a smooth function on the boundary. For $t > 0$ (the elliptic part) we tried to apply the method using the Calderon projection (see Boutet de Monvel [3] and Chazarain/Piriou [4]). Unfortunately, our effort to construct solutions with this method broke down on the fact that the method brought with it too many technical problems we could not solve. Therefore we will only consider the simple boundary $t = t_0$ is constant (see section 4.5).

4.2. The Cauchy problem.

The first problem we discuss is

$$(4.2.1)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} + t\dfrac{\partial^2 u}{\partial x^2} = 0, \\ u|_{\Gamma_0} = f, \\ \dfrac{\partial u}{\partial t}\Big|_{\Gamma_0} = g. \end{cases}$$

Here $\Gamma_0 = \{(x,t) \mid t = 0\} \subset \mathbb{R}^2$.

For $t < 0$ this is a (degenerate) hyperbolic initial value problem. In case of $f \in C^1$ and $g \in C^0$ a classical solution has been obtained by constructing a Riemann function for T. This solution is given by

$$(4.2.2)\qquad \gamma_1 \int_{-1}^{+1} f(x + \tfrac{2}{3}(-t)^{\frac{3}{2}}s)[1 - s^2]^{-\frac{5}{6}}ds + \gamma_2 t \int_{-1}^{+1} g(x + \tfrac{2}{3}(-t)^{\frac{3}{2}}s)[1 - s^2]^{-\frac{1}{6}}ds,$$

with

$$\gamma_1 = \frac{2^{\frac{2}{3}}\Gamma(\frac{1}{3})}{\Gamma^2(\frac{1}{6})} \quad\text{and}\quad \gamma_2 = \frac{2^{-\frac{2}{3}}\Gamma(\frac{5}{3})}{\Gamma^2(\frac{5}{6})}.$$

See Bizadse [2].

This solution can be obtained in another way. Assume for the moment $f \in C_0^\infty(\mathbb{R})$ and $g \in C_0^\infty(\mathbb{R})$. Application of Fourier transformation with respect to x gives for $v(t) := \tilde{u}(\xi,t)$ the equations

$$\frac{d^2v}{dt^2} - t|\xi|^2 v = 0, \quad v(0) = \hat{f}(\xi) \text{ and } \frac{dv}{dt}(0) = \hat{g}(\xi).$$

This is solved by

$$(4.2.3)\qquad \tilde{u}(\xi,t) = c_1(\xi)\mathrm{Ai}(t|\xi|^{\frac{2}{3}}) + c_2(\xi)\mathrm{Bi}(t|\xi|^{\frac{2}{3}}),$$

with

$$c_1(\xi) = \pi(\mathrm{Bi}'(0)\hat{f} - \frac{1}{|\xi|^{\frac{2}{3}}}\mathrm{Bi}(0)\hat{g}),$$

$$c_2(\xi) = -\pi(\mathrm{Ai}'(0)\hat{f} - \frac{1}{|\xi|^{\frac{2}{3}}}\mathrm{Ai}(0)\hat{g}), \quad \xi \neq 0.$$

For the functions Ai and Bi see section A.2.
We have $\frac{1}{2}(\mathrm{Ai}(z) \pm i\mathrm{Bi}(z)) = e^{\pm\pi i/3}\mathrm{Ai}(ze^{\mp 2\pi i/3})$, so we can rewrite $\tilde{u}$ as

$$\frac{1}{2\pi i}\tilde{u}(\xi,t) = \mathrm{Ai}'(0)\Big[e^{\pi i/3}\mathrm{Ai}(t|\xi|^{\frac{2}{3}}e^{2\pi i/3}) - e^{-\pi i/3}\mathrm{Ai}(t|\xi|^{\frac{2}{3}}e^{-2\pi i/3})\Big]\hat{f}(\xi)$$

$$+ \mathrm{Ai}(0)\Big[\frac{1}{|\xi|^{\frac{2}{3}}}(\mathrm{Ai}(t|\xi|^{\frac{2}{3}}e^{2\pi i/3}) - \mathrm{Ai}(t|\xi|^{\frac{2}{3}}e^{-2\pi i/3}))\Big]\hat{g}(\xi).$$

For $t \le 0$ $Ai(t|\xi|^{\frac{2}{3}}e^{2\pi i/3})$ and $Ai(t|\xi|^{\frac{2}{3}}e^{-2\pi i/3})$ are bounded, so we can apply Fourier's inversion formula. Define for $t \le 0$

$$U_t := \frac{1}{2\pi}\int e^{ix\xi}\left[e^{\pi i/3}Ai(t|\xi|^{\frac{2}{3}}e^{2\pi i/3}) - e^{-\pi i/3}Ai(t|\xi|^{\frac{2}{3}}e^{-2\pi i/3})\right]d\xi,$$

$$V_t := \frac{1}{2\pi}\int e^{ix\xi}\frac{1}{|\xi|^{\frac{2}{3}}}\left[Ai(t|\xi|^{\frac{2}{3}}e^{2\pi i/3}) - Ai(t|\xi|^{\frac{2}{3}}e^{-2\pi i/3})\right]d\xi .$$

LEMMA 4.2.4.

1. U_t *and* V_t *have support contained in* $\{x \mid |x| \le \frac{2}{3}(-t)^{\frac{3}{2}}\}$.
2. U_t *and* V_t *are elements of* $C^\infty(\overline{\mathbb{R}_t^-}, \mathcal{E}'(\mathbb{R}_x))$.
3. *If* $\varphi_j \to 0$ *in* $C^\infty(\mathbb{R}_x)$ *then* $\langle U_t,\varphi_j\rangle \to 0$ *in* $C^\infty(\overline{\mathbb{R}_t^-})$. *A similar statement holds for* V_t.

PROOF. 1. The powerseries of $Ai(z)$ (see section A.2) shows that $\hat{U}_t$ and $\hat{V}_t$ can be continued analytically to an entire function in ξ. Moreover, the asymptotic expansion for $Ai(z)$ shows that these functions are bounded on $\mathbb{C}$ by a polynomial times $\exp \frac{2}{3}(-t)^{\frac{3}{2}} |\mathrm{Im}\ \xi|$. So the result follows from the Paley-Wiener Theorem.

2. Lemma A.5.2 shows that for $-T \le t \le 0$

$$(4.2.5) \qquad \forall\ n \ge 0: \exists\ C_n: \left|\frac{\partial^n}{\partial t^n}Ai(t|\xi|^{\frac{2}{3}}e^{\pm 2\pi i/3})\right| \le C_n(1+|\xi|)^{2n}.$$

So for $\varphi \in C_0^\infty$, in $\langle U_t,\varphi\rangle = \langle \hat{U}_t,\check{\varphi}\rangle$ we can differentiate under the integral sign, since $\check{\varphi}$ is rapidly decreasing. For $\varphi \in C^\infty$ the result follows by using an appropriate cut-off function.

3. Without restriction $\varphi_j \to 0$ in $C_0^\infty(\mathbb{R}_x)$. Then $D_t^n\langle U_t,\varphi_j\rangle = D_t^n\langle \hat{U}_t,\check{\varphi}_j\rangle$. We have:

$$\forall\ m: \exists\ (C_m^j)_j: |\check{\varphi}_j(\xi)| \le C_m^j(1+|\xi|)^{-m} \text{ and } C_m^j \to 0\ (j \to \infty).$$

Together with estimate (4.2.5) this shows that $D_t^n\langle U_t,\varphi_j\rangle \to 0$ uniformly on $[-T,0]$, T arbitrary. □

For $t \le 0$, $f \in \mathcal{D}'(\mathbb{R}_x)$ and $g \in \mathcal{D}'(\mathbb{R}_x)$ we now define

$$(4.2.6) \qquad E(f,g)(\cdot,t) := 2\pi i[Ai'(0)U_t * f + Ai(0)V_t * g] .$$

The convolutions are with respect to x only. They are welldefined because of Lemma 4.2.4.

PROPOSITION 4.2.7.

1. $\forall$ f,g $\in \mathcal{D}'(\mathbb{R})$: $E(f,g) \in C^\infty(\overline{\mathbb{R}_t^-}, \mathcal{D}'(\mathbb{R}_x))$.
2. $E(f,g)$ *depends continuously on* (f,g).
3. *The restrictions of* $E(f,g)$ *and* $\frac{\partial}{\partial t}E(f,g)$ *to* Γ_0 *are welldefined.*
4. $E(f,g)$ *can be considered as an element of* $\mathcal{D}'(\mathbb{R}\times\mathbb{R}^-)$, *still depending continuously on* (f,g). *It is a solution of problem* (4.2.1) *for* $t < 0$.

PROOF. 1. $\langle U_t * f,\varphi\rangle = \langle U_t(x),\langle f(y),\varphi(x+y)\rangle\rangle$ is smooth for $t \le 0$, $\varphi \in C_0^\infty(\mathbb{R})$, because $\langle f(y),\varphi(x+y)\rangle$ is smooth in x.

2. If $f_j \to 0$ in $\mathcal{D}'(\mathbb{R})$ then $\langle f_j(y),\varphi(x+y)\rangle \to 0$ in $C^\infty(\mathbb{R}_x)$. So we can apply Lemma 4.2.4, part 3.

3. This follows from the first part.

4. For $\psi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^-)$ $\langle E(f,g),\psi\rangle := \int_{-\infty}^0 \langle E(f,g)(\cdot,t),\psi(\cdot,t)\rangle dt$.
This defines a distribution on $\mathbb{R}\times\mathbb{R}^-$ and from
$\langle U_t * f,\psi\rangle = \int \langle U_t(x),\langle f(y),\psi(x+y,t)\rangle\rangle dt$
it is clear that this distribution depends continuously on f and g.
For f and g in $C_0^\infty(\mathbb{R})$ $E(f,g)$ obviously defines a solution for problem (4.2.1). Now $C_0^\infty(\mathbb{R})$ is dense in $\mathcal{D}'(\mathbb{R})$ and the restriction operator (in the sense of part 3) is continuous, so the result follows by continuity. □

Let us now examine the singularities of E(f,g). From estimate (4.2.5) it is clear that E(f,g) is smooth in (x,t), $t \le 0$ for $f,g \in C_0^\infty(\mathbb{R})$ and so for $f,g \in C^\infty(\mathbb{R})$ also. We recall that

$$Ai(t|\xi|^{\frac{2}{3}}e^{\pm 2\pi i/3}) = e^{\pm\frac{2}{3}i(-t)^{\frac{3}{2}}|\xi|}\, a_\pm(t,\xi),\ \xi \ne 0,\ t < 0.$$

See section A.5. Here $\omega(\xi)a_\pm(t,\xi)$ is an element of $S_{1,0}^{-\frac{1}{6}}$ if $\omega(\xi)$ is a smooth function such that

$$\omega(\xi) = \begin{cases}1 & \text{for } |\xi| \ge 2\\ 0 & \phantom{\text{for }} |\xi| \le 1\end{cases}.$$

If we substitute this in the definition (4.2.6) of E(f,g) we get for $f \in C_0^\infty$, $g \in C_0^\infty$, $t < 0$:

$$(4.2.8)\qquad E(f,g)(x,t) \equiv \int_{-\infty}^{\infty} d\xi e^{ix\xi}\frac{\omega(\xi)}{|\xi|^{\frac{2}{3}}} iAi(0)\ \times$$
$$\left\{\left[e^{\pi i/3}\frac{Ai'(0)}{Ai(0)}|\xi|^{\frac{2}{3}}\hat{f}(\xi) + \hat{g}(\xi)\right]a_+(t,\xi)e^{\frac{2}{3}i(-t)^{\frac{3}{2}}|\xi|}\right.$$
$$\left.-\left[e^{-\pi i/3}\frac{Ai'(0)}{Ai(0)}|\xi|^{\frac{2}{3}}\hat{f}(\xi) + \hat{g}(\xi)\right]a_-(t,\xi)e^{-\frac{2}{3}i(-t)^{\frac{3}{2}}|\xi|}\right\}$$

modulo operators with smooth kernels (cf. section 2.8, example 3 and section 3.4).

It is clear that this expression can be considered as a sum of FIOs with elliptic symbols and non-degenerate phase functions $\varphi_{\pm} := (x-y)\xi \pm \frac{2}{3}(-t)^{\frac{3}{2}}|\xi|$. These phase functions are just the functions $\varphi_{(\pm,+)}$ defined in section 3.2 with s = 0.

By continuity equation (4.2.8) holds for f,g in $\mathcal{E}'(\mathbb{R})$, too, and for $f,g \in \mathcal{D}'(\mathbb{R})$ as well because we can make the symbols to be properly supported.

The relations defined by these phase functions are

$$\Lambda'_{\varphi_{\pm}} = \{(y \mp \tfrac{2}{3}(-t)^{\frac{3}{2}}\frac{\xi}{|\xi|}, t, \xi, \mp(-t)^{\frac{1}{2}}|\xi|; y, \xi) \mid t < 0, \xi \neq 0\}.$$

So a FIO with phase function $(x-y)\xi + \frac{2}{3}(-t)^{\frac{3}{2}}|\xi|$ propagates a singularity in (y,ξ) to the left if $\xi > 0$ and to the right if $\xi < 0$. For the other phase function the other way round.

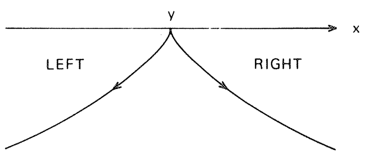

Fig. 9: propagation in (x,t)-space.

Indeed singularities are propagated along bicharacteristic strips. Lemma 4.2.4 shows that $E(f,g)(x_0,t_0)$ only depends on the data on that part of the boundary cut out by the bicharacteristic curves through (x_0,t_0). The same is represented by expression (4.2.2). If we express the Airy functions in terms of Bessel functions and use the integral expression (A.1.7), it is not difficult to show that for f and g in $C_0^{\infty}(\mathbb{R})$ expressions (4.2.2) and (4.2.6) are equal.

<u>REMARK 4.2.9</u>. Let $t_1 < 0$, $u_1 := E(f,g)\big|_{t=t_1}$ and $v_1 := \frac{\partial}{\partial t}E(f,g)\big|_{t=t_1}$. Of course, these restrictions are welldefined. u_1 and v_1 are the Cauchy data of E(f,g) on $t = t_1$, while f and g are the Cauchy data on t = 0. From formula (4.2.8) it follows that modulo smooth functions u_1 and v_1 are given by

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.$$

Here A_{11}, A_{12}, A_{21} and A_{22} are sums of FIOs with elliptic symbols of order $-\frac{1}{6}$, $-\frac{5}{6}$, $\frac{5}{6}$ and $\frac{1}{6}$, respectively.
For a strictly hyperbolic Cauchy problem it is wellknown that the corresponding operators are sums of FIOs with symbols of order 0, −1, 1 and 0, respectively. Note that T is strictly hyperbolic only for $t < 0$. If we solve the Cauchy problem for T with data on $t = t_0 < 0$, we obtain an expression for the solution similar to expression (4.2.8). We use the same cut-off function ω. And indeed, restriction to $t = t_1 < t_0$ then gives sums of FIOs with symbols of order 0, −1, 1 and 0, respectively. These symbols depend on t_0, t_1 and ξ. Keeping t_1 fixed, it is not difficult to show that for $t_0 \uparrow 0$ these symbols converge to the corresponding symbols of the operators A_{11}, A_{12}, A_{21} and A_{22} in $S^0_{1,0}$, $S^{-\frac{5}{6}+\varepsilon}_{1,0}$, $S^1_{1,0}$ and $S^{\frac{1}{6}+\varepsilon}_{1,0}$, respectively. Here $\varepsilon > 0$ is arbitrary.

REMARK 4.2.10. $\hat{U}_t$ and $\hat{V}_t$ only depend on t and $|\xi|$. Moreover, for $x \in \mathbb{R}^n$ we have

$$\left[\left(\frac{\partial^2}{\partial t^2} + t\Delta_x\right)u\right]^{\sim} = \left(\frac{\partial^2}{\partial t^2} - t|\xi|^2\right)\tilde{u}.$$

The operator $\partial^2/\partial t^2 - t|\xi|^2$ only depends on t and $|\xi|$, too. From this it follows easily that by interpreting x as $(x_1,\dots,x_n)$, ξ as $(\xi_1,\dots,\xi_n)$ and $x\xi$ as $\langle x,\xi\rangle$, formula (4.2.6) gives a solution to the problem

$$\frac{\partial^2 u}{\partial t^2} + t\Delta_x u = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}^-,$$

$$u\big|_{\Gamma_0} = f,$$

$$\frac{\partial u}{\partial t}\Big|_{\Gamma_0} = g.$$

Interpreted as elements of $\mathcal{D}'(\mathbb{R}^n_x \times \mathbb{R}^-_t)$ U_t and V_t only depend on t and $|x|$, have support in $\{x \in \mathbb{R}^n \mid |x| \le \frac{2}{3}(-t)^{\frac{3}{2}}\}$ and are singular only for $|x| = \frac{2}{3}(-t)^{\frac{3}{2}}$. The singularities have order $(|x| - \frac{2}{3}(-t)^{\frac{3}{2}})^{-\frac{1}{3}-n/2}$ and $(|x| - \frac{2}{3}(-t)^{\frac{3}{2}})^{\frac{1}{3}-n/2}$, respectively.

This concludes the discussion for $t < 0$.

For $t > 0$ we remark that in formula (4.2.3) the function $Bi(t|\xi|^{\frac{2}{3}})$ is exponentially increasing for $|\xi| \to \infty$. Suppose we are interested in values of t such that $0 \le t \le T$, $0 < T \le \infty$. In order to be able to apply

Fourier's inversion formula we will assume for some C^{∞}-function $\chi(\xi)$ which is zero in a neighbourhood of $\xi = 0$ and one for $|\xi|$ large:

$$c_2(\xi)\chi(\xi)e^{\frac{2}{3}|\xi|T^{\frac{3}{2}}} \in S'(\mathbb{R}) \quad \text{if } T < \infty,$$
$$c_2(\xi)\chi(\xi) = 0 \qquad \text{if } T = \infty.$$

For the problem (4.2.1) on $\mathbb{R} \times (0,T)$ this means that we cannot prescribe f and g arbitrarily. It implies that

$$(4.2.11) \qquad (\chi\hat{g})(\xi) = \frac{|\xi|^{\frac{2}{3}}}{\pi \mathrm{Ai}(0)}\hat{h}(\xi)e^{-\frac{2}{3}|\xi|T^{\frac{3}{2}}} + \frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)}|\xi|^{\frac{2}{3}}(\chi\hat{f})(\xi)$$

for some $h \in S'$, $h = 0$ for $T = \infty$.

For such g we define for $t \le T$ ($t < \infty$ if $T = \infty$):

$$(4.2.12)\ E_{\chi}^{T}(f,g)(x,t) = \int d\xi e^{ix\xi} \times$$
$$\Big\{ i\mathrm{Ai}'(0)\Big[e^{\pi i/3}\mathrm{Ai}(t|\xi|^{\frac{2}{3}}e^{2\pi i/3}) - e^{-\pi i/3}\mathrm{Ai}(t|\xi|^{\frac{2}{3}}e^{-2\pi i/3})\Big](1-\chi(\xi))\hat{f}(\xi)$$
$$+ i\mathrm{Ai}(0)\Big[-\frac{1}{|\xi|^{\frac{2}{3}}}(\mathrm{Ai}(t|\xi|^{\frac{2}{3}}e^{2\pi i/3}) - \mathrm{Ai}(t|\xi|^{\frac{2}{3}}e^{-2\pi i/3}))\Big](1-\chi(\xi))\hat{g}(\xi)$$
$$+ \frac{1}{2\pi \mathrm{Ai}(0)}\mathrm{Ai}(t|\xi|^{\frac{2}{3}})(\chi\hat{f})(\xi)$$
$$- \frac{1}{2\pi \mathrm{Ai}(0)}\Big[\mathrm{Bi}(0)\mathrm{Ai}(t|\xi|^{\frac{2}{3}}) - \mathrm{Ai}(0)\mathrm{Bi}(t|\xi|^{\frac{2}{3}})\Big]e^{-\frac{2}{3}|\xi|T^{\frac{3}{2}}}\hat{h}(\xi)\Big\} .$$

Here h is defined by equation (4.2.11).
In the first two terms $\hat{f}$ and $\hat{g}$ are multiplied by smooth functions with compact support. In the last two terms $\chi\hat{f}$ and $\hat{h}$ are zero in a neighbourhood of $\xi = 0$, so multiplications are welldefined for f and h in S'.

<u>PROPOSITION 4.2.13</u>. *Let* χ *be as above,* $f \in S'(\mathbb{R})$, $g \in S'(\mathbb{R})$ *so that for some* h g *satisfies condition* (4.2.11). *Then*:

1. $E_{\chi}^{T}(f,g) \in C^{\infty}((-\infty,T]_t, S'(\mathbb{R}_x))$. *Here* $(-\infty,T] := \mathbb{R}$ *if* $T = \infty$.
2. *If* $f_j \to 0$ *in* S', $g_j \to 0$ *in* S' *and there are* h_j *so that* g_j *satisfies condition* (4.2.11) *with* h_j *and* $h_j \to 0$ *in* S' *then* $E_{\chi}^{T}(f_j,g_j) \to 0$ (*as element of* $C^{\infty}((-\infty,T],S'(\mathbb{R}))$).
3. $E_{\chi}^{T}(f,g)$ *is a smooth function on* $\mathbb{R} \times (0,T)$.
4. *The restrictions of* $E_{\chi}^{T}(f,g)$ *and* $\frac{\partial}{\partial t}E_{\chi}^{T}(f,g)$ *to* Γ_0 *are welldefined.*
5. $E_{\chi}^{T}(f,g)$ *can be considered as an element of* $\mathcal{D}'(\mathbb{R} \times (-\infty,T))$, *satisfying the same continuity property as element of* $\mathcal{D}'$ *as in part* 2. *It is a solution of problem* (4.2.1) *for* $t < T$ (*with a special choice of* g!).

PROOF. 1,2. By writing $<E_\chi^T(f,g),\varphi>$ in terms of Fourier transforms, this is a simple verification using Lemma A.5.2.

3. This follows from formula (4.2.12) and Lemma A.5.2.

4. This follows from the first part.

5. As in Proposition 4.2.7 for $\psi \in C_0^\infty(\mathbb{R} \times (-\infty,T))$

$$<E_\chi^T(f,g),\psi> := \int_{-\infty}^{T} <E_\chi^T(f,g)(\cdot,t),\psi(\cdot,t)>dt.$$

This is continuous in (f,g) as can be seen again by expressing it in terms of Fourier transforms.
For f and g in $C_0^\infty(\mathbb{R})$ it defines a solution of problem (4.2.1) in $\mathbb{R} \times (-\infty,T)$. For general f and g it follows by continuity. □

REMARK 4.2.14. If g satisfies condition (4.2.11) for some (χ,h) then for all χ_1 with the same properties as χ there is h_1 such that g satisfies condition (4.2.11) for (χ_1,h_1). Moreover $E_\chi^T(f,g)$ is independent of the choice for (χ,h). Therefore we can omit the χ-sign in case $T < \infty$.

REMARK 4.2.15. $E^T(f,g)$ is equal to $E(f,g)$ on $\mathbb{R} \times \mathbb{R}^-$. The singularities of $E^T(f,g)$ for $t \le 0$ are therefore known.

REMARK 4.2.16. We can rewrite condition (4.2.11) as

$$\int e^{i<x-y,\xi>} \frac{Ai'(0)}{Ai(0)}|\xi|^{\frac{2}{3}}\chi(\xi)f(y)dyd\xi - \int e^{i<x-y,\xi>}\chi(\xi)g(y)dyd\xi = 0.$$

This is a Pseudo Differential relation between f and g. For simplicity we neglect the other term. The symbols have order $\frac{2}{3}$ and 0, respectively. For regular elliptic problems one gets a similar relation between the Cauchy data, however with symbols of order 1 and 0, respectively. This relation is determined by the Calderon projection (see Chazarain/Piriou [4]). Note that T is elliptic only for $t > 0$. Solving the Cauchy problem for T with data on $t = t_0 > 0$, indeed we obtain a relation similar to the relation given above with symbols of order 1 and 0, respectively. These symbols converge to the symbols given above in $S^1_{1,0}$, $S^0_{1,0}$, respectively for $t_0 \downarrow 0$.

For later convenience we give one more property of the distributions U_t and V_t discussed in Lemma 4.2.4.

LEMMA 4.2.17. *For fixed* $t \le 0$ U_t *can be written as the sum of two* *distributions with support contained in* $\{x \mid x \ge \frac{2}{3}(-t)^{\frac{3}{2}}\}$ *and* $\{x \mid x \ge -\frac{2}{3}(-t)^{\frac{3}{2}}\}$ *respectively. Also* U_t *can be written for fixed* $t \le 0$ *as the sum of two distributions with support contained in* $\{x \mid x \le \frac{2}{3}(-t)^{\frac{3}{2}}\}$ *and* $\{x \mid x \le -\frac{2}{3}(-t)^{\frac{3}{2}}\}$ *respectively. A similar statement holds for* V_t.

PROOF.
$$e^{\pi i/3}Ai(t|\xi|^{\frac{2}{3}}e^{2\pi i/3}) - e^{-\pi i/3}Ai(t|\xi|^{\frac{2}{3}}e^{-2\pi i/3}) =$$
$$= e^{\pi i/3}Ai(t(\pm\xi)^{\frac{2}{3}}e^{2\pi i/3}) - e^{-\pi i/3}Ai(t(\pm\xi)^{\frac{2}{3}}e^{-2\pi i/3}) \quad \text{for } \xi \in \mathbb{R},\ \xi \neq 0$$

and the terms are analytic for $\mathrm{Im}\ \xi \gtrless 0$.
We claim that
$$\mathrm{supp}[Ai(t\xi^{\frac{2}{3}}e^{\pm 2\pi i/3})]^{\vee} \subset \{x \mid x \le \mp \tfrac{2}{3}(-t)^{\frac{3}{2}}\}$$
and
$$\mathrm{supp}[Ai(t(-\xi)^{\frac{2}{3}}e^{\pm 2\pi i/3})]^{\vee} \subset \{x \mid x \ge \pm \tfrac{2}{3}(-t)^{\frac{3}{2}}\}.$$
Let us show this for $[Ai(t(-\xi)^{\frac{2}{3}}e^{2\pi i/3})]^{\vee}$.
It is clear that it is a welldefined element of $\mathcal{S}'(\mathbb{R})$, for $Ai(t(-\xi)^{\frac{2}{3}}e^{2\pi i/3})$ is bounded and continuous for ξ real. The support can be computed directly (cf. the argument above Remark 4.2.9), but we can also note that for $\mathrm{Im}\,\xi < 0$
$$e^{\frac{2}{3}i(-t)^{\frac{3}{2}}\xi}Ai(e^{2\pi i/3}t(-\xi)^{\frac{2}{3}})$$

is a bounded analytic function, continuous for $\mathrm{Im}\ \xi \le 0$, for $|\arg e^{2\pi i/3}t(-\xi)^{\frac{2}{3}}| \le \frac{\pi}{3}$ then. Further
$$\int e^{ix\xi}Ai(e^{2\pi i/3}t(-\xi)^{\frac{2}{3}})d\xi = \left(1 - \frac{\partial^2}{\partial x^2}\right)\int e^{ix\xi}Ai(e^{2\pi i/3}t(-\xi)^{\frac{2}{3}})\frac{d\xi}{1+\xi^2}.$$
An application of complex contour integration shows that for $x - \frac{2}{3}(-t)^{\frac{3}{2}} < 0$ this is equal to
$$\left(1 - \frac{\partial^2}{\partial x^2}\right)\left[-e^{x}Ai(-t)\right] = 0.$$
The distribution V_t can be treated in a similar way. □

4.3. The Goursat problem.

Let
$$\Omega^{-} := \{(x,t) \mid x > 0 \text{ and } -\left(\frac{3x}{2}\right)^{\frac{2}{3}} < t < 0\},$$
$$\Gamma_{+} := \{(x,t) \mid x > 0 \text{ and } t = 0\},$$
$$\Gamma := \{(x,t) \mid x > 0 \text{ and } t = -\left(\frac{3x}{2}\right)^{\frac{2}{3}}\}.$$

Then Γ is part of a bicharacteristic curve and bnd $\Omega^- = \mathrm{cl}(\Gamma_+ \cup \Gamma)$.

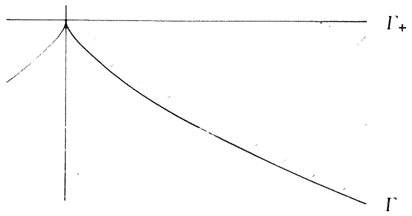

Fig. 10: Ω^-.

We choose x as the parameter along Γ. Then Γ can be identified with $\mathbb{R}^+$. Consider now the problem

$$(4.3.1)\qquad \begin{cases} Tu = 0 \text{ in } \Omega^-, \\ u|_{\Gamma_+} = f, \\ u|_{\Gamma} = h. \end{cases}$$

We will take $h \in E'(\Gamma)$ (so that $0 \notin \mathrm{supp}(h)$) and $f \in M_\alpha$ for some $\alpha > \frac{5}{6}$. Here M_α is defined by $M_\alpha := \{u \in \mathcal{D}'(\mathbb{R}) \mid \mathrm{supp}(u) \subset \overline{\mathbb{R}^+}$ and $\exists\, \varphi \in C_0^\infty(\mathbb{R})$: $\varphi(x) = 1$ in a neighbourhood of $x = 0$ and $\varphi u \in H^\alpha(\mathbb{R})\}$, where H^α is a Sobolev space. Note that the condition $f \in M_\alpha$ for $\alpha > \frac{5}{6}$ implies that f is continuous in $x = 0$ (in particular $f(0) = 0$).
The reason for this choice for f is that M_α is large enough to be useful in section 4.5 and small enough to be useful in this section.

The problem will be solved by determining g such that $E(f,g)|_\Gamma = h$. That is, we must determine $g \in \mathcal{D}'(\mathbb{R})$ such that the restriction of $E(f,g)$ to Γ is welldefined and equal to h. Note that Γ is characteristic.
First we will discuss the case $h = 0$.
Then it is obvious that we should choose g so that $E(f,g)$ will not have singularities on bicharacteristic curves going to the left. Formula (4.2.8) and the discussion following it show that probably a good choice for g will be

$$(4.3.2)\qquad \hat{g}(\xi) := -\frac{Ai'(0)}{Ai(0)}|\xi|^{\frac{2}{3}}e^{\pm\pi i/3}\hat{f}(\xi) \text{ for } \xi \gtrless 0.$$

This equals

$$-\frac{Ai'(0)}{Ai(0)}e^{-\pi i/3}(-\xi + i0)^{\frac{2}{3}}\hat{f}(\xi),$$

so

$$g(x) = -\frac{1}{\Gamma(-\frac{2}{3})}\frac{Ai'(0)}{Ai(0)} x_+^{-\frac{5}{3}} * f(x).$$

For arbitrary f with $supp(f) \subset \overline{\mathbb{R}^+}$ this convolution is welldefined and $supp(g) \subset \overline{\mathbb{R}^+}$, too.

Define

$$Ef := E(f,g) \text{ with g chosen as above.}$$

Then $T(Ef) = 0$ for $t < 0$ and $Ef|_{\Gamma_+} = f$.

To show that Ef satisfies the boundary condition on Γ is more difficult.

An inspection of formulas (4.2.6) and (4.2.8) shows that for $f \in C_0^\infty(\mathbb{R})$, Ef is given by:

$$\begin{aligned} Ef &= -Ai'(0)\sqrt{3}\Big[\int_{-\infty}^{0} d\xi e^{ix\xi}\hat{f}(\xi)Ai(t|\xi|^{\frac{2}{3}}e^{2\pi i/3}) \\ &\qquad + \int_{0}^{\infty} d\xi e^{ix\xi}\hat{f}(\xi)Ai(t|\xi|^{\frac{2}{3}}e^{-2\pi i/3})\Big] \\ &= -Ai'(0)\sqrt{3}\int_{-\infty}^{\infty} d\xi e^{ix\xi}\hat{f}(\xi)Ai(e^{2\pi i/3}t(-\xi)^{\frac{2}{3}}). \end{aligned}$$

<u>PROPOSITION 4.3.3</u>. *The map* $\mathcal{D}'_+ \ni f \to Ef \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^-)$ *is continuous. For* $f \in \mathcal{D}'_+$, Ef *is given by convolution of* $f \otimes \delta_{t=0}$ *and a distribution with support in*

$$\{(x,t) \mid t < 0 \text{ and } x \geq \tfrac{2}{3}(-t)^{\frac{3}{2}}\}.$$

<u>PROOF</u>. If $f_j \to f$ in $\mathcal{D}'_+$ then $x_+^{-\frac{5}{3}} * f_j \to x^{-\frac{5}{3}} * f$ in $\mathcal{D}'_+$. E(f,g) depends continuously on f and g. So $f \to Ef$ is continuous. From Lemma 4.2.17 it is clear that

$$\int e^{ix\xi}Ai(t(-\xi)^{\frac{2}{3}}e^{2\pi i/3})d\xi$$

is a distribution in $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^-)$ with support in $\{(x,t) \mid x \geq \frac{2}{3}(-t)^{\frac{3}{2}}\}$.
For $f \in C_0^\infty(\mathbb{R})$ it is clear that Ef is given by convolution of $f \otimes \delta_{t=0}$ with this distribution. This convolution is welldefined for $f \in \mathcal{D}'_+$ as well and continuously depending on f. But then the continuity of E shows that Ef is given by this convolution for arbitrary $f \in \mathcal{D}'_+$. □

Modulo a smoothing operator we can also describe Ef by means of a FIO.

$$(4.3.4) \qquad Ef \equiv -Ai'(0)\sqrt{3}\int_{-\infty}^{\infty} e^{ix\xi-\frac{2}{3}i(-t)^{\frac{3}{2}}\xi}a(t,\xi)\hat{f}(\xi)d\xi.$$

Here $a(t,\xi) = \omega(\xi)[H(\xi)a_-(t,\xi) + H(-\xi)a_+(t,\xi)]$ is a symbol of order $-\frac{1}{6}$

for $t < 0$ which is elliptic and $(x-y-\frac{2}{3}(-t)^{\frac{3}{2}})\xi$ is a nondegenerate phase function for $t < 0$.
This is obvious for $f \in C_0^\infty(\mathbb{R})$. For $f \in \mathcal{E}'(\mathbb{R})$ it follows by continuity.

PROPOSITION 4.3.5.

1. *Let* $f \in M_\alpha$ *for some* $\alpha > \frac{1}{3}$. *Then the restriction of* Ef *to* Γ *is well-defined.*
2. *Let* $f \in M_\alpha$ *for some* $\alpha > \frac{5}{6}$ *and let* γ *be the trace operator on* Γ. *Then* $Ef|_\Gamma = \gamma(Ef) = 0$.

PROOF. We can write $f = f_1 + f_2$ with $f_1 \in H^\alpha_{comp}(\mathbb{R})$, $supp(f_1) \subset \overline{\mathbb{R}^+}$ and $f_2 \in \mathcal{D}'(\mathbb{R})$, $supp(f_2) \subset \mathbb{R}^+$. Let $\varepsilon > 0$ be so that $supp(f_2) \subset [\varepsilon,\infty)$. Then $supp(Ef_2) \subset \{(x,t) \mid x-\frac{2}{3}(-t)^{\frac{3}{2}} \geq \varepsilon\}$ so we can assume $f_2 = 0$, that is $f = f_1$.
The FIO given by equation (4.3.4) has order $-\frac{5}{12}$, since $n_1 = 2$, $n_2 = 1$, $m = -\frac{1}{6}$ and $N = 1$. See section 2.8, paragraph (2.8.7).
Now Corollary 4.4.5 in Duistermaat [6] shows that E is continuous from $H^s_{comp}(\mathbb{R})$ to $H^{s+\frac{1}{6}}_{loc}(\mathbb{R}\times\mathbb{R}^-)$ for every $s \in \mathbb{R}$. So $Ef \in H^{\alpha+\frac{1}{6}}_{loc}(\mathbb{R}\times\mathbb{R}^-)$. From the theory of Sobolev spaces it is wellknown that the trace operator γ is continuous $H^{\alpha+\frac{1}{6}}_{loc}(\mathbb{R}\times\mathbb{R}^-) \to H^{\alpha-\frac{1}{3}}_{loc}(\Gamma)$, provided $\alpha+\frac{1}{6} > \frac{1}{2}$ or $\alpha > \frac{1}{3}$. This concludes the first part.
If $\alpha > \frac{5}{6}$ and $f \in H^\alpha_{comp}$, then $Ef \in H^{\alpha+\frac{1}{6}}_{loc}(\mathbb{R}\times\mathbb{R}^-)$. So Ef is continuous for $t < 0$ because $\alpha+\frac{1}{6} > 1$. Also $supp(f) \subset \overline{\mathbb{R}^+}$ so Proposition 4.3.3 shows that $supp(Ef) \subset \overline{\Omega^-}$. A combination of these facts gives the desired result. □

REMARK 4.3.6. The fact that we must choose $\alpha > \frac{5}{6}$ is sufficient for the application of Proposition 4.3.5 in section 4.5. However, note that $\alpha > \frac{5}{6}$, $f \in H^\alpha$ implies f is continuous. In particular, $f(0) = 0$. The converse is not true. So we do not allow every continuous function which is zero in $x = 0$. This is in contrast with the work of other authors. See for instance Bizadse [2]. Presumably the fact that we work with H^α-spaces is to blame for this, together with the fact that we did not compute the distribution in Proposition 4.3.3 explicitly.
If we keep $t = t_0 < 0$ fixed and consider E as an operator from M_α to $\mathcal{D}'(\mathbb{R}\times\{t_0\})$, then $f \in H^\alpha_{comp}$ implies $Ef|_{t=t_0} \in H^{\alpha+\frac{1}{6}}_{comp}$. This is a continuous function in x provided $\alpha > \frac{1}{3}$. So we get an indication that the result of Proposition 4.3.5 might be improved.

Next we consider the case $f = 0$.
If $g \in \mathcal{D}'_0(\mathbb{R}^+)$, then there exists a unique $\tilde{g} \in \mathcal{D}'(\mathbb{R})$ such that $\tilde{g}|_{\mathbb{R}^+} = g$ and

$\mathrm{supp}(\tilde{g}) = \mathrm{supp}(g)$.
Define $E_0 g := E(0,\tilde{g})$.
The restriction of $E_0 g$ to Γ is welldefined. This follows from the fact that $WF(E_0 g) \cap N(\Gamma) = \emptyset$ because $\tilde{g}$ is smooth in a neighbourhood of $x = 0$ (see section 2.6).

PROPOSITION 4.3.7. $g \to \gamma(E_0 g)$ *determines a continuous map between* $\mathcal{D}_0'(\mathbb{R}^+)$ *and* $\mathcal{D}_0'(\Gamma)$.

PROOF. The map is welldefined and $\gamma(E_0 g) \in \mathcal{D}_0'(\Gamma)$. For the proof of the continuity it is sufficient to consider for all $\varepsilon > 0$ the restriction of this map to $\mathcal{E}'(Y_\varepsilon)$, where $Y_\varepsilon = \{y \in \mathbb{R}^+ \mid y > \varepsilon\}$. Let $(u_j)_j \subset \mathcal{E}'(Y_\varepsilon)$ be a sequence so that $u_j \to u$ $(j \to \infty)$, $u \in \mathcal{E}'(Y_\varepsilon)$. Then $E_0 u_j \to E_0 u$ in $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^-)$. We will show that the sequence $(E_0 u_j)_j$ converges to $E_0 u$ in $\mathcal{D}'_{K_\varepsilon}$ with K_ε defined by

$$K_\varepsilon := \{(x,t,\xi,\tau) \mid x - \tfrac{2}{3}(-t)^{\frac{3}{2}} \geq \tfrac{1}{2}\varepsilon \text{ or } \tau = -(-t)^{\frac{1}{2}}\xi\}.$$

This is a closed cone in $T^*(\mathbb{R}_x \times \mathbb{R}_t^-)$ and $WF(E_0 v) \subset K_\varepsilon$ if $v \in \mathcal{E}'(Y_\varepsilon)$.
Let A be a properly supported ΨDO with $WF(A) \cap K_\varepsilon = \emptyset$.
Then AE_0 has a C^∞-kernel. This follows from formula (2.7.8) in section 2.7.
So $AE_0 : \mathcal{E}'(Y_\varepsilon) \to C^\infty(\mathbb{R} \times \mathbb{R}^-)$ continuously and

$$A(E_0 u_j) = (AE_0)u_j \to (AE_0)u = A(E_0 u) \text{ in } C^\infty(\mathbb{R} \times \mathbb{R}^-).$$

The remark made in section 2.9 shows that $E_0 u_j \to E_0 u$ in $\mathcal{D}'_{K_\varepsilon}$.

Now $K_\varepsilon \cap N(\Gamma) = \emptyset$, so the result given in section 2.6 shows that $\gamma(E_0 u_j) \to \gamma(E_0 u)$ in $\mathcal{D}'(\Gamma)$ and also in $\mathcal{D}_0'(\Gamma)$. □

We define $Sg := \gamma(E_0 g)$. For $g \in C_0^\infty(\mathbb{R}^+)$ Sg is given by:

$$(4.3.8) \qquad (Sg)(x) = iAi(0) \int d\xi e^{ix\xi} \frac{1}{|\xi|^{\frac{2}{3}}}\Big[Ai(-(\tfrac{3x}{2})^{\frac{2}{3}}|\xi|^{\frac{2}{3}}e^{2\pi i/3}) - Ai(-(\tfrac{3x}{2})^{\frac{2}{3}}|\xi|^{\frac{2}{3}}e^{-2\pi i/3})\Big]\hat{g}(\xi).$$

Substitution of $t = -(\frac{3x}{2})^{\frac{2}{3}}$ in formula (4.2.8) produces the phase functions $(2x-y)\xi$ and $-y\xi$. Operators related to the last phase function have kernels with wave front set contained in $\{(x,0;0,-\xi) \mid \xi \neq 0\}$. Therefore these operators are smoothing on $\mathcal{D}_0'$, for $u \in \mathcal{D}_0'$ implies u is smooth in $x = 0$.
So for $g \in \mathcal{D}_0'$:

$$Sg = \gamma(E(0,\tilde{g})) \equiv iAi(0) \iint dyd\xi e^{i(2x-y)\xi} \times$$

$$\frac{\omega(\xi)}{|\xi|^{\frac{2}{3}}}\left[H(\xi)a_+(-(\tfrac{3x}{2})^{\frac{2}{3}},\xi) - H(-\xi)a_-(-(\tfrac{3x}{2})^{\frac{2}{3}},\xi)\right]g(y).$$

That is, S is an elliptic FIO modulo a smoothing operator. Therefore S has a parametrix. It is even rather easy to give a proper inverse for S. We can transform formula (4.3.8) into a formula that can also be derived from formula (4.2.2):

Consider $g \in C_0^\infty(\mathbb{R}^+)$ and formula (4.3.8).

$$Ai(-(\tfrac{3x}{2})^{\frac{2}{3}}|\xi|^{\frac{2}{3}}e^{2\pi i/3}) - Ai(-(\tfrac{3x}{2})^{\frac{2}{3}}|\xi|^{\frac{2}{3}}e^{-2\pi i/3}) =$$

$$\tfrac{1}{2}i\sqrt{3}\ Ai\left(-(\tfrac{3x}{2})^{\frac{2}{3}}|\xi|^{\frac{2}{3}}\right) - \tfrac{1}{2}iBi\left(-(\tfrac{3x}{2})^{\frac{2}{3}}|\xi|^{\frac{2}{3}}\right) =$$

$$\frac{i}{\sqrt{3}}(\tfrac{3x}{2})^{\frac{1}{3}}|\xi|^{\frac{1}{3}}J_{\frac{1}{3}}(x|\xi|)$$

and

$$J_{\frac{1}{3}}(x|\xi|) = \frac{(\frac{1}{2}x|\xi|)^{\frac{1}{3}}}{\Gamma(\frac{5}{6})\Gamma(\frac{1}{2})}\int_{-1}^{+1}(1-s^2)^{-\frac{1}{6}}e^{ix|\xi|s}ds.$$

See sections A.1 and A.2.

Here we can omit the absolute sign in $e^{ix|\xi|s}$. But then

$$Sg = -Ai(0)\frac{(\frac{1}{2}x)^{\frac{1}{3}}(3x/2)^{\frac{1}{3}}}{\sqrt{3}\Gamma(\frac{5}{6})\Gamma(\frac{1}{2})}\int d\xi e^{ix\xi}\hat{g}(\xi)\int_{-1}^{+1}ds(1-s^2)^{-\frac{1}{6}}e^{ix\xi s}.$$

Changing the order of integration we get

$$Sg = -2\pi Ai(0)\frac{3^{-\frac{5}{6}}(3x/2)^{\frac{2}{3}}}{\Gamma(\frac{5}{6})\Gamma(\frac{1}{2})}\int_{-1}^{+1}ds(1-s^2)^{-\frac{1}{6}}g(x+sx).$$

Here

$$\frac{2\pi Ai(0)}{3^{\frac{5}{6}}\Gamma(\frac{5}{6})\Gamma(\frac{1}{2})} = \gamma_2.$$

Compare this to formula (4.2.2).

Substituting $x(1+s) = s'$,

$$(4.3.9) \qquad (Sg)(x) = -\gamma_2(\tfrac{3}{2})^{\frac{2}{3}}\int_0^{2x}\frac{g(s)ds}{s^{\frac{1}{6}}(2x-s)^{\frac{1}{6}}}.$$

Equating this to h(x) we get Abel's equation for $s_+^{-\frac{1}{6}}g(s)$.

In terms of Fourier transforms it can be solved as follows. We must solve:

$$-\gamma_2(\tfrac{3}{2})^{\frac{2}{3}}x_+^{-\frac{1}{6}}g * x_+^{-\frac{1}{6}} = h(\tfrac{x}{2}).$$

Then

$$-\gamma_2(\tfrac{3}{2})^{\frac{2}{3}}(x_+^{-\frac{1}{6}}g)^\wedge \cdot \Gamma(\tfrac{5}{6})e^{5\pi i/12}(-\xi + i0)^{-\frac{5}{6}} = 2\hat{h}(2\xi).$$

So

$$(x_+^{-\frac{1}{6}}g)^\wedge = 2(-\gamma_2(\tfrac{3}{2})^{\frac{2}{3}}\Gamma(\tfrac{5}{6})e^{5\pi i/12})^{-1}(-\xi + i0)^{+\frac{5}{6}}\hat{h}(2\xi).$$

For the distributions x_+^α and $(-\xi + i0)^\alpha$, see section 2.13.

Define

$$(4.3.10)\qquad Qh := (-\gamma_2(\tfrac{3}{2})^{\frac{2}{3}}\Gamma(\tfrac{5}{6})e^{5\pi i/12})^{-1}x_+^{\frac{1}{6}}\frac{1}{\pi}\int e^{ix\xi}(-\xi + i0)^{+\frac{5}{6}}\hat{h}(2\xi)d\xi.$$

PROPOSITION 4.3.11. *Q can be extended to a continuous map between* $\mathcal{D}_0'(\mathbb{R}^+)$ *and* $\mathcal{D}_0'(\mathbb{R}^+)$. *Then* SQh = h *for every* $h \in \mathcal{D}_0'(\mathbb{R}^+)$.

PROOF. In fact Q represents convolution of $h(\frac{x}{2})$ and a distribution with support in $\overline{\mathbb{R}^+}$, for $[(-\xi + i0)^{\frac{5}{6}}]^\vee = \text{const}\cdot x_+^{-\frac{11}{6}}$. This extends continuously to $h \in \mathcal{D}_0'(\mathbb{R}^+)$ giving a distribution which is zero in a neighbourhood of the origin. Therefore multiplication with $x_+^{\frac{1}{6}}$ is welldefined.
It is evident that SQh = h for $h \in C_0^\infty(\mathbb{R}^+)$. Now Proposition 4.3.7 shows that S is continuous and so is Q. But then SQh = h for every $h \in \mathcal{D}_0'(\mathbb{R}^+)$. □

REMARK 4.3.12. Formula (4.3.10) shows that Q is an elliptic FIO modulo a smoothing operator. Its phase function is $(x-2y)\xi$.

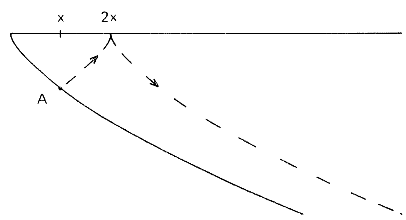

Fig. 11: propagation of a singularity of h in A.

We can now define: $E_-(f,h) := Ef + E(0,Qh)$.

PROPOSITION 4.3.13. $E_-(f,h)$ *is a solution for problem* (4.3.1). *If* $f_j \in \cup_{\alpha>\frac{5}{6}} M_\alpha$ $\forall\, j \geq 0$ *so that* $f_j \to f_0$ *in* $\mathcal{D}'(\mathbb{R})$ *and if* $h_j \to h_0$ *in* $\mathcal{D}_0'(\mathbb{R}^+)$, *then* $E_-(f_j,h_j) \to E_-(f_0,h_0)$ *in* $\mathcal{D}'(\Omega^-)$.

PROOF. That $E_-(f,h)$ is a solution for problem (4.3.1) is clear. The continuity follows from Proposition 4.3.3 and Proposition 4.3.11. □

REMARK 4.3.14. Formula (4.3.4) and Remark (4.3.12) show that singularities in f and h are propagated to the right (cf. Fig. 11).

4.4. A non-characteristic boundary value problem.

Let $\varphi(x)$ be a C^∞-function for $x \geq 0$, $\varphi(0) = 0$, $\varphi(x) > 0$ for $x > 0$ and $-1 < \varphi^{\frac{1}{2}}(x)\frac{d\varphi}{dx}(x) < 1$. Let Γ_+ be as in section 4.3 and let Γ be the curve parametrized by $\Gamma = \{(x,-\varphi(x)) \mid x > 0\}$. Again Γ can be identified with $\mathbb{R}^+$ by choosing x as parameter along Γ. Note that Γ is nowhere characteristic. Consider now the region $\Omega := \{(x,t) \mid x > 0 \text{ and } -\varphi(x) < t < 0\}$.

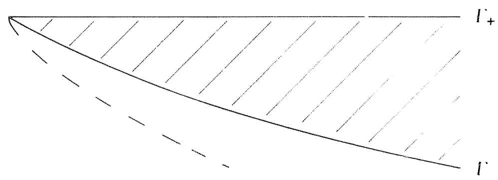

Fig. 12: Ω.

We will investigate the problem

$$(4.4.1) \quad \begin{cases} Tu = 0 & \text{in } \Omega, \\ u|_{\Gamma_+} = f, & \\ u|_{\Gamma} = h & \end{cases}$$

with f and h in $\mathcal{D}_0'(\mathbb{R}^+)$.
Again we will try to solve this problem by looking for a g such that $E(f,g)|_\Gamma = h$. Note that for arbitrary $g \in \mathcal{D}'(\mathbb{R})$ the restriction of $E(f,g)$ to Γ is welldefined because Γ is nowhere characteristic.
However, it turns out that we can only give a solution for problem (4.4.1) modulo a smooth function on Γ, that is: $u|_\Gamma - h$ is smooth. This is due to the fact that we did not succeed in obtaining a proper inverse for some elliptic ΨDO.
For f,g in $\mathcal{D}_0'(\mathbb{R}^+)$ we define operators A_1 and A_2 by

$$A_1 g := (2\pi i Ai(0) V *_x g)|_\Gamma,$$
$$A_2 f := (2\pi i Ai'(0) U *_x f)|_\Gamma.$$

For $U = U_t$ and $V = V_t$ see formula (4.2.6). U and V are considered as elements of $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^-)$.

PROPOSITION 4.4.2. *A_1 and A_2 are continuous maps from $\mathcal{D}_0'(\Gamma^+)$ to $\mathcal{D}_0'(\Gamma)$. The kernel of A_1 in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^+)$ has support contained in $\{(x,y) \mid x > 0$ and $q(x) \leq y \leq p(x)\}$. Here $q(x) := x - \frac{2}{3}\varphi^{\frac{3}{2}}(x)$ and*

$p(x) := x + \frac{2}{3}\varphi^{\frac{3}{2}}(x)$ *are smooth and monotonically increasing functions from* $\mathbb{R}^+$ *to* $\mathbb{R}^+$.

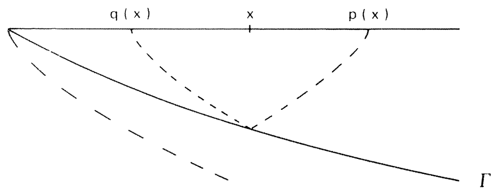

Fig. 13: interrupted lines are bicharacteristic curves.

So A_1 *is properly supported.*

PROOF. That $A_1 g$ and $A_2 f$ belong to $\mathcal{D}'_0(\Gamma)$ follows from the properties of the supports of V and U.
If $g_j \to 0$ in $\mathcal{D}'_0(\mathbb{R}^+)$ then $E(0,g_j) \to 0$ in $\mathcal{D}'_K(\mathbb{R} \times \mathbb{R}^-)$ with $K = \{(x,t,\xi,\tau) \mid t|\xi|^2 + \tau^2 = 0\}$. Since $K \cap N(\Gamma) = \emptyset$, $A_1 g_j \to 0$ in $\mathcal{D}'(\Gamma)$ and also in $\mathcal{D}'_0(\Gamma)$, so A_1 is continuous.
For $\psi \in C_0^\infty(\mathbb{R}_x)$, $g \in C_0^\infty(\mathbb{R}_y)$: $\langle (V * g)|_\Gamma, \psi\rangle \overset{!}{=} \langle\langle V_{-\varphi(x)}(y), g(x-y)\rangle, \psi(x)\rangle$ so the kernel of A_1 has support in $\{(x,y) \mid |x-y| \le \frac{2}{3}\varphi^{\frac{3}{2}}(x),\ x > 0\}$.
The properties of q and p follow from the properties of φ. □

The results of section 4.2 show that for $g \in C_0^\infty(\mathbb{R}^+)$ $A_1 g$ is given by

$$(A_1 g)(x) = iAi(0) \int_{-\infty}^{\infty} d\xi e^{ix\xi} \frac{1}{|\xi|^{\frac{2}{3}}} \Big[Ai(-\varphi(x)|\xi|^{\frac{2}{3}} e^{2\pi i/3}) - Ai(-\varphi(x)|\xi|^{\frac{2}{3}} e^{-2\pi i/3}) \Big] \hat{g}(\xi).$$

Here g is extended by $g(x) = 0$ for $x \le 0$.

Also Lemma 4.2.17 shows that A_1 can be written as $A_1 = B_1 + B_2$ with B_1 and B_2 having kernels $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively so that $\operatorname{supp} \mathcal{B}_1 \subset \{(x,y) \mid p(x) \ge y\}$ and $\operatorname{supp} \mathcal{B}_2 \subset \{(x,y) \mid q(x) \ge y\}$. If $\operatorname{supp}(g) \subset [x_0, x_1]$, $0 < x_0 < x_1 < \infty$, then $\operatorname{supp}(A_1 g) \subset [p^{-1}(x_0), q^{-1}(x_1)]$, $\operatorname{supp}(B_1 g) \subset [p^{-1}(x_0), \infty)$ and $\operatorname{supp}(B_2 g) \subset [q^{-1}(x_0), \infty)$.

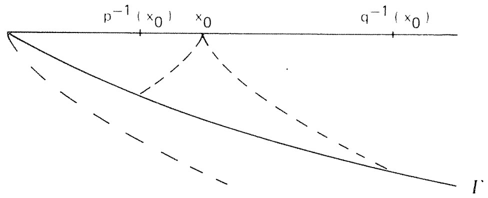

Fig. 14: interrupted lines are bicharacteristic curves.

Suppose B_1^{-1} exists and has the property supp $B_1^{-1}g \subset [p(x_0),\infty)$. Then supp $B_1^{-1}B_2 g \subset [pq^{-1}(x_0),\infty)$. So the support of g will shift to infinity by repeated application of $B_1^{-1}B_2$ (see also Lemma 4.4.10). But then

$$\sum_{n=0}^{\infty} (-B_1^{-1}B_2)^n B_1^{-1}$$

is a welldefined inverse for A_1.
A similar result holds if B_2 is invertible. Then the support of g will shift to zero.
Unfortunately we are not able to obtain an exact inverse for B_1 (or B_2). Therefore we restrict ourselves to the determination of a parametrix for A_1. Modulo smoothing operators we will write A_1 as $A_1 \equiv C_1 + C_2$. Here C_1 and C_2 can be chosen to be properly supported elliptic FIOs. C_1 and C_2 will be related to phase functions $(p(x)-y)\xi$ and $(q(x)-y)\xi$ respectively and the support of their kernels will have similar properties as the support of B_1 and B_2.

Let $\omega(\xi)$ be a C^∞-function on $\mathbb{R}$ such that

$$\omega(\xi) = \begin{cases} 0 & \text{for } |\xi| < \frac{1}{2} \\ 1 & \phantom{\text{for }} |\xi| > 1 \end{cases}.$$

We know that

$$Ai(-\varphi(x)|\xi|^{\frac{2}{3}}e^{2\pi i/3}) = e^{\frac{2}{3}i\varphi^{\frac{3}{2}}(x)|\xi|}\, a_+(-\varphi(x),\xi),$$
$$Ai(-\varphi(x)|\xi|^{\frac{2}{3}}e^{-2\pi i/3}) = e^{-\frac{2}{3}i\varphi^{\frac{3}{2}}(x)|\xi|}\, a_-(-\varphi(x),\xi),\ \xi \neq 0,$$

and $|\xi|^{-\frac{2}{3}}\omega(\xi)a_\pm(-\varphi(x),\xi)$ are elements of $S_{1,0}^{-\frac{5}{6}}(\mathbb{R}^+ \times \mathbb{R})$ which are elliptic (see Lemma A.5.1).

Let $\chi(x,y)$ be a C^∞-function on $\mathbb{R}^+ \times \mathbb{R}^+$ so that $\chi = 1$ on a neighbourhood of the support of the kernel of A_1 and so that χ is properly supported. For $g \in C_0^\infty(\mathbb{R}^+)$ then $\langle V_{-\varphi(x)}(y), g(x-y)\rangle = \langle V_{-\varphi(x)}(y), \chi(x,x-y)g(x-y)\rangle$

because $1-\chi(x,x-y)$ is zero on a neighbourhood of $\mathrm{supp}(V_{-\varphi(x)})$.
Fourier transformation gives that

$$(A_1 g)(x) = iAi(0) \int d\xi e^{ix\xi} \frac{1}{|\xi|^{\frac{2}{3}}} \times \left[Ai(-\varphi(x)|\xi|^{\frac{2}{3}} e^{2\pi i/3}) - Ai(-\varphi(x)|\xi|^{\frac{2}{3}} e^{-2\pi i/3})\right] \times \int dy e^{-iy\xi}\chi(x,y)g(y).$$

Let χ_1 and χ_2 be smooth and properly supported functions on $\mathbb{R}^+ \times \mathbb{R}^+$ which are equal to one in a neighbourhood of the diagonal in $\mathbb{R}^+ \times \mathbb{R}^+$ and choose

$$(4.4.3) \qquad \chi(x,y) = \begin{cases} \chi_2(q(x),y) & y \le q(x) \\ 1 & \text{if } q(x) \le y \le p(x) \\ \chi_1(p(x),y) & y \ge p(x) \end{cases}.$$

Then it is easy to check that χ satisfies the conditions mentioned before.

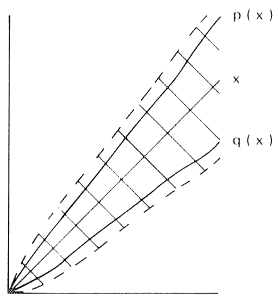

Fig. 15: support of χ.

Define

$$(A_{11}g)(x) = \iint dyd\xi e^{i(x-y+\frac{2}{3}\varphi^{\frac{3}{2}}(x))\xi} a_{11}(x,\xi)\chi_1(p(x),y)g(y),$$

$$(A_{12}g)(x) = \iint dyd\xi e^{i(x-y-\frac{2}{3}\varphi^{\frac{3}{2}}(x))\xi} a_{12}(x,\xi)\chi_2(q(x),y)g(y),$$

$$(4.4.4) \qquad (A_{13}g)(x) = \iint dyd\xi \Big[e^{i(x-y+\frac{2}{3}\varphi^{\frac{3}{2}}(x))\xi} a_{11}(x,\xi)[\chi(x,y)-\chi_1(p(x),y)]g(y) + e^{i(x-y-\frac{2}{3}\varphi^{\frac{3}{2}}(x))\xi} a_{12}(x,\xi)[\chi(x,y)-\chi_2(q(x),y)]g(y)\Big],$$

$$(A_{14}g)(x) = iAi(0) \int dyg(y)\chi(x,y) \int d\xi e^{i(x-y)\xi}[1-\omega(\xi)] \frac{1}{|\xi|^{\frac{2}{3}}} \times \left[Ai(-\varphi(x)|\xi|^{\frac{2}{3}} e^{2\pi i/3}) - Ai(-\varphi(x)|\xi|^{\frac{2}{3}} e^{-2\pi i/3})\right].$$

Here

$$a_{11}(x,\xi) = iAi(0)\omega(\xi)\frac{1}{|\xi|^{\frac{2}{3}}}[H(\xi)a_+(-\varphi(x),\xi) - H(-\xi)a_-(-\varphi(x),\xi)] \ ,$$
$$a_{12}(x,\xi) = iAi(0)\omega(\xi)\frac{1}{|\xi|^{\frac{2}{3}}}[H(-\xi)a_+(-\varphi(x),\xi) - H(\xi)a_-(-\varphi(x),\xi)] \ .$$

Then it is clear that a_{11} and a_{12} belong to $S^{-\frac{5}{6}}_{1,0}(\mathbb{R}^+ \times \mathbb{R})$ and are elliptic. A_{11}, A_{12} and A_{13} are welldefined (sums of) properly supported FIOs. So they define continuous maps between $\mathcal{D}'(\mathbb{R}^+)$ and $\mathcal{D}'(\mathbb{R}^+)$ (even $C^\infty(\mathbb{R}^+)$ and $C^\infty(\mathbb{R}^+)$). Note that $\chi(x,y) - \chi_1(p(x),y)$ is zero in a neighbourhood of the set of critical points of $(p(x) - y)\xi$ and $\chi(x,y) - \chi_2(q(x),y)$ is zero in a neighbourhood of the set of critical points of $(q(x) - y)\xi$. So A_{13} is smoothing. The ξ-integral in the definition of A_{14} is a smooth function on $\mathbb{R}^+ \times \mathbb{R}^+$. Therefore A_{14} defines a continuous map from $\mathcal{D}'(\mathbb{R}^+)$ to $C^\infty(\mathbb{R}^+)$.

PROPOSITION 4.4.5. $A_1 = A_{11} + A_{12} + A_{13} + A_{14}$ *on* $\mathcal{D}'_0(\Gamma_+)$.

PROOF. The operators are continuous on $\mathcal{D}'_0(\Gamma_+)$, so we can restrict ourselves to showing that $A_1 g = \Sigma^4_{j=1} A_{1j}g$ for $g \in C^\infty_0(\mathbb{R}^+)$. For fixed x the continuous dependence of A_{1j}, $j = 1,2,3$, on their symbol(s) shows that $(A_{1j}g)(x) = \lim_{\varepsilon \downarrow 0} (A^\varepsilon_{1j}g)(x)$, where A^ε_{1j} is obtained from A_{1j} by replacing $\omega(\xi)$ by $\alpha(\varepsilon\xi)\omega(\xi)$, with $\alpha \in S$, $\alpha(0) = 1$.
But then the proof amounts to an application of Fubini's theorem. □

The operators A_{11} and A_{12} will play the role of C_1 and C_2 in the remark made above. For this purpose we will construct suitable cut-off functions χ_1 and χ_2.

LEMMA 4.4.6. *Define for* $x > 0$, $0 < a \leq 1$ *the functions* c_a *and* d_a *by*

$$c_a(x) := x - \frac{2a}{3}\varphi^{\frac{3}{2}}(x) \ \textit{and} \ d_a(x) := x + \frac{2a}{3}\varphi^{\frac{3}{2}}(x).$$

Then c_a *and* d_a *are* C^∞ *for* $x > 0$ *and for* $a < 1$
$q(x) = c_1(x) < c_a(x) < x < d_a(x) < d_1(x) = p(x)$.
Further $1-a < c'_a(x) < 1+a$ *and* $1-a < d'_a(x) < 1+a$.

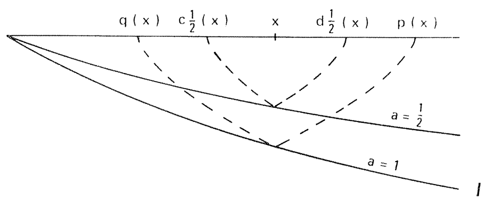

Fig. 16: interrupted lines are bicharacteristic curves.

PROOF. The properties of c_a and d_a follow easily from the properties of φ. □

For $0 < b < a < 1$ let $\chi_1^{(a,b)}$ and $\chi_2^{(a,b)}$ be elements of $C^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$ so that

$$\chi_1^{(a,b)}(x,y) = \begin{cases} 0 & \text{if } y \le c_a(x) \text{ or } y \ge d_a(x) \\ 1 & \text{if } c_b(x) \le y \le d_b(x) \end{cases},$$

$$\chi_2^{(a,b)}(x,y) = \begin{cases} 0 & \text{if } y \le d_a^{-1}(x) \text{ or } y \ge c_a^{-1}(x) \\ 1 & \text{if } d_b^{-1}(x) \le y \le c_b^{-1}(x) \end{cases}.$$

These are functions which are properly supported and equal to one in a neighbourhood of the diagonal in $\mathbb{R}^+ \times \mathbb{R}^+$. Note that in order to keep this neighbourhood small the parameter a should be chosen small. In that case the functions c_a, d_a, c_b and d_b are "almost" the identity.

LEMMA 4.4.7. *Let* $g \in C_0^\infty(\mathbb{R}^+)$ *and* $\mathrm{supp}(g) \subset [x_0,x_1]$. *Then*:

$$\mathrm{supp}\left(\int \chi_1^{(a,b)}(x,y)g(y)dy\right) \subset [d_a^{-1}(x_0), c_a^{-1}(x_1)] ,$$

$$\mathrm{supp}\left(\int \chi_2^{(a,b)}(x,y)g(y)dy\right) \subset [c_a(x_0), d_a(x_1)] . \qquad \square$$

For χ_1 and χ_2 we will take the functions $\chi_1^{(a,b)}$ and $\chi_2^{(a,b)}$ for some (a,b).

Let us consider now the operator A_{11}.
If we substitute $z = p(x)$ we obtain the ΨDO

$$\iint dyd\xi e^{i(z-y)} a_{11}(p^{-1}(z),\xi)\chi_1(z,y)g(y).$$

Here $a_{11}(p^{-1}(z),\xi)\chi_1(z,y)$ is a properly supported element of $S_{1,0}^{-\frac{5}{6}}$ which is elliptic.
So this operator has a properly supported parametrix. Modulo a smoothing operator it can be given by

$$\psi \to \iint dyd\xi e^{i(x-y)} \chi_1(x,y) f_{11}(x,\xi)\psi(y)$$

for some f_{11} belonging to $S_{1,0}^{\frac{5}{6}}$. A parametrix for A_{11} is then given by

$$(F_{11}\psi)(x) = \iint dyd\xi e^{i(x-p(y))\xi} \chi_1(x,p(y)) f_{11}(x,\xi)p'(y)\psi(y).$$

LEMMA 4.4.8. $A_{11}F_{11} - I = R_1$ *and* $F_{11}A_{11} - I = R_2$, *where* R_1 *and* R_2 *have* C^∞ *kernels so that*

$$\mathrm{supp}(\text{kernel of } R_1) \subset \{(x,y) \mid c_a c_a p(x) \le p(y) \le d_a d_a p(x)\},$$

$$\mathrm{supp}(\text{kernel of } R_2) \subset \{(x,y) \mid c_a c_a(x) \le y \le d_a d_a(x)\}. \qquad \square$$

LEMMA 4.4.9. *Let* $g \in C_0^\infty(\mathbb{R}^+)$ *so that* $\mathrm{supp}\, g \subset [x_0, x_1]$ *and* $a < 1$. *With* $a_0 = \frac{1-a}{1+a}$ *and* $a_1 = \frac{1-a}{2}$ *then:*

$$\mathrm{supp}(F_{11}A_{12}g) \subset [d_a^{-1}pq^{-1}c_a(x_0), c_a^{-1}pq^{-1}d_a(x_1)] \subset [d_{a_0}(x_0), \infty),$$

$$\mathrm{supp}(A_{12}F_{11}g) \subset [q^{-1}c_a d_a^{-1}p(x_0), q^{-1}d_a c_a^{-1}p(x_1)] \subset [d_{a_1}(x_0), \infty),$$

$$\mathrm{supp}({}^tA_{12}\,{}^tF_{11}g) \subset [d_a^{-1}qp^{-1}c_a(x_0), c_a^{-1}qp^{-1}d_a(x_1)] \subset (0, c_{a_0}(x_1)],$$

$$\mathrm{supp}({}^tF_{11}\,{}^tA_{12}g) \subset [p^{-1}c_a d_a^{-1}q(x_0), p^{-1}d_a c_a^{-1}q(x_1)] \subset (0, c_{a_1}(x_1)].$$

PROOF. The kernel of ${}^tA_{12}$ has support in $\{(x,y) \mid \chi_2(q(y),x) \ne 0\}$. The kernel of ${}^tF_{11}$ has support in $\{(x,y) \mid \chi_1(y,p(x)) \ne 0\}$. Therefore the first inclusions follow from Lemma 4.4.7.
From Lemma 4.4.6:

$$q^{-1}c_a(y) - y = q^{-1}c_a(y) - q^{-1}q(y) > \tfrac{1}{2}(c_a(y) - q(y)) = \tfrac{1}{3}(1-a)\varphi^{\frac{3}{2}}(y),$$

$$d_a^{-1}p(y) - y = d_a^{-1}p(y) - d_a^{-1}d_a(y) > \frac{1}{1+a}(p(y) - d_a(y)) = \frac{1-a}{1+a}\,\frac{2}{3}\varphi^{\frac{3}{2}}(y),$$

$$y - c_a^{-1}q(y) = c_a^{-1}c_a(y) - c_a^{-1}q(y) > \frac{1}{1+a}(c_a(y) - q(y)) = \frac{1-a}{1+a}\,\frac{2}{3}\varphi^{\frac{3}{2}}(y),$$

$$y - p^{-1}d_a(y) = p^{-1}p(y) - p^{-1}d_a(y) > \tfrac{1}{2}(p(y) - d_a(y)) = \tfrac{1}{3}(1-a)\varphi^{\frac{3}{2}}(y),$$

so

$$d_a^{-1}pq^{-1}c_a(x_0) - x_0 > d_a^{-1}p(x_0) - x_0 > \frac{1-a}{1+a}\,\frac{2}{3}\varphi^{\frac{3}{2}}(x_0),$$

$$q^{-1}c_a d_a^{-1}p(x_0) - x_0 > q^{-1}c_a(x_0) - x_0 > \tfrac{1}{3}(1-a)\varphi^{\frac{3}{2}}(x_0),$$

$$c_a^{-1}qp^{-1}d_a(x_1) - x_1 < c_a^{-1}q(x_1) - x_1 < -\frac{1-a}{1+a}\,\frac{2}{3}\varphi^{\frac{3}{2}}(x_1),$$

$$p^{-1}d_a c_a^{-1}q(x_1) - x_1 < p^{-1}d_a(x_1) - x_1 < -\tfrac{1}{3}(1-a)\varphi^{\frac{3}{2}}(x_1). \qquad \square$$

LEMMA 4.4.10. *Let* $0 < a \le 1$ *and choose* $z > 0$. *Define*

$$z_n := d_a^n(z) \quad (d_a \textit{ applied } n \textit{ times})$$

$$w_n := c_a^n(z).$$

Then $z_n \to \infty$ *and* $w_n \downarrow 0$ *for* $n \to \infty$.

PROOF. $\varphi(x) > 0$ for $x > 0$ and $a > 0$ so $(z_n)_n$ is monotonically increasing. Suppose (z_n) is bounded. Then $z_n \to z_\infty < \infty$. But then $d_a(z_n) - z_n = z_{n+1} - z_n \to 0$. Because d_a is continuous it follows that $d_a(z_\infty) = z_\infty$ or

$\varphi^{\frac{3}{2}}(z_\infty) = 0$. This is a contradiction. So (z_n) is not bounded and $z_n \to \infty$. $(w_n)_n$ is monotonically decreasing and bounded by 0. If $w_n \to w_0$ then $w_n - c_a(w_n) = w_n - w_{n+1} \to 0$ and the continuity of c_a gives that $c_a(w_0) = w_0$ or $\varphi^{\frac{3}{2}}(w_0) = 0$. This implies that $w_0 = 0$. □

At this point we know enough about the supports to define a parametrix for A_1.
For $x > 0$ and $g \in C_0^\infty(\mathbb{R}^+)$ we define

$$(4.4.11) \qquad (Gg)(x) := \Big(\sum_{n=0}^{\infty} (-F_{11}A_{12})^n F_{11}g\Big)(x).$$

We also consider the transpose of G:

$$(4.4.12) \qquad ({}^tGg)(x) := \Big(\sum_{n=0}^{\infty} {}^tF_{11}(-{}^tA_{12}\,{}^tF_{11})^n g\Big)(x).$$

At this point we recall the definitions of $\mathcal{E}_0$, $\mathcal{E}_\infty$, $\mathcal{D}'_0$ and $\mathcal{D}'_\infty$ given in section 2.1.

PROPOSITION 4.4.13.

1. *Equation* (4.4.11) *defines a continuous map between* $C_0^\infty(\mathbb{R}^+)$ *and* $\mathcal{E}_0$. *Equation* (4.4.12) *defines a continuous map between* $C_0^\infty(\mathbb{R}^+)$ *and* $\mathcal{E}_\infty$.
2. $A_1G = I + T_1$ *and* $GA_1 = I + T_2$ *with*

$$T_1 = [R_1 + (A_{13} + A_{14})F_{11}] \sum_{n=0}^{\infty} (-A_{12}F_{11})^n,$$
$$T_2 = \sum_{n=0}^{\infty} (-F_{11}A_{12})^n [R_2 + F_{11}(A_{13} + A_{14})] .$$

PROOF. 1. Lemmas 4.4.9 and 4.4.10 show that on every compact $K \subset \mathbb{R}^+$ the series defining Gg and ${}^tGg$ in fact are finite (so are the series defining T_1 and T_2). That $Gg \in \mathcal{E}_0$ follows from the fact that $A_{12}g$ and $F_{11}g$ belong to $C_0^\infty(\mathbb{R}^+)$. Similarly ${}^tGg \in \mathcal{E}_\infty$.
The continuity of G and ${}^tG$ follows from the continuity of A_{12}, ${}^tA_{12}$, F_{11} and ${}^tF_{11}$.
2.

$$\begin{aligned} A_1G &= (A_{11} + A_{12} + A_{13} + A_{14})\Big(\sum_{n=0}^{\infty} F_{11}(-A_{12}F_{11})^n\Big) \\ &= -\sum_{n=0}^{\infty} (-A_{12}F_{11})^{n+1} + (I + R_1) \sum_{n=0}^{\infty} (-A_{12}F_{11})^n \\ &\qquad + (A_{13} + A_{14}) \sum_{n=0}^{\infty} F_{11}(-A_{12}F_{11})^n \\ &= I + [R_1 + (A_{13} + A_{14})F_{11}] \sum_{n=0}^{\infty} (-A_{12}F_{11})^n. \end{aligned}$$

$$GA_1 = \sum_{n=0}^{\infty} (-F_{11}A_{12})^n F_{11}(A_{11} + A_{12} + A_{13} + A_{14})$$
$$= \sum_{n=0}^{\infty} (-F_{11}A_{12})^n (I + R_2) - \sum_{n=0}^{\infty} (-F_{11}A_{12})^{n+1} +$$
$$+ \sum_{n=0}^{\infty} (-F_{11}A_{12})^n F_{11}(A_{13} + A_{14})$$
$$= I + \sum_{n=0}^{\infty} (-F_{11}A_{12})^n [R_2 + F_{11}(A_{13} + A_{14})] \, .$$ □

REMARK 4.4.14. It is clear that G can be defined on E_0 as well. Then $G: E_0 \xrightarrow{\text{continuous}} E_0$ and Proposition 4.4.13 (2) remains valid due to the fact that the operators have properly supported kernels.
Also ${}^tG: E_\infty \xrightarrow{\text{continuous}} E_\infty$.

By transposition Gu can now be defined for $u \in \mathcal{D}'_0$ and ${}^tGu$ for $u \in \mathcal{D}'_\infty$ by

(4.4.15)
$$\langle Gu,\varphi\rangle := \langle u, {}^tG\varphi\rangle,$$
$$\langle {}^tGu,\varphi\rangle := \langle u,G\varphi\rangle, \; \varphi \in C_0^\infty(\mathbb{R}^+).$$

For u smooth this coincides with definitions (4.4.11) and (4.4.12).

PROPOSITION 4.4.16.

1. G *is a continuous map between* $\mathcal{D}'_0$ *and* $\mathcal{D}'_0$. ${}^tG$ *is a continuous map between* $\mathcal{D}'_\infty$ *and* $\mathcal{D}'_\infty$.
2. $A_1Gu - u$ *and* $GA_1u - u$ *belong to* E_0.

PROOF. 1. This is straightforward.
2. This follows from the fact that T_1 and T_2 map $\mathcal{D}'_0$ continuously to E_0 because R_1, R_2, A_{13} and A_{14} are (properly supported) smoothing operators. □

We return to the boundary value problem (4.4.1)
Define

$$E_2(f,h) := E(f,G(h - A_2f)).$$

PROPOSITION 4.4.17. $E_2(f,h)$ *is a solution of problem* (4.4.1) *modulo a smooth function on* Γ.

PROOF.
$$E_2(f,h)|_\Gamma = A_2f + A_1G(h - A_2f) = A_2f + h - A_2f + T_1(h - A_2f) =$$
$$= h + T_1(h - A_2f)$$

and

$$T_1(h - A_2f) \in E_0.$$ □

Let us now discuss the way singularities of $E_2(f,h)$ are related to those of f and h.

LEMMA 4.4.18. *For* $u \in \mathcal{D}_0'(\mathbb{R}^+)$:

$$WF(Gu) \subset \{((pq^{-1})^n p(y),\xi) \mid n \in \mathbb{Z},\ n \geq 0,\ (y,\xi) \in WF(u)\}.$$

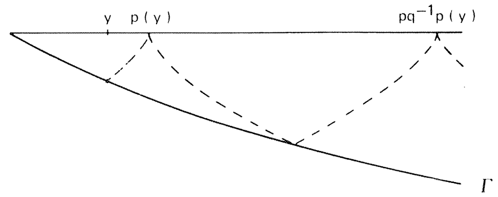

Fig. 17: interrupted lines are bicharacteristic curves.

PROOF. The phase functions of F_{11} and A_{12} are $(x-p(y))\xi$ and $(q(x)-y)\xi$. The symbols of F_{11} and A_{12} are elliptic for $|\xi|$ large. Therefore

$$WF(F_{11}u) = \{(p(y),\xi) \mid (y,\xi) \in WF(u)\},$$

$$WF(A_{12}u) = \{(q^{-1}(y),\xi) \mid (y,\xi) \in WF(u)\}.$$

Composition gives the desired result. □

For $f = 0$, $E_2(0,h) = E(0,Gh)$. If we take into consideration the way E propagates singularities we can conclude that singularities in h on Γ are propagated to the right, possibly with reflexion at Γ_+ or Γ.
For $h = 0$, $E_2(f,0) = E(f,-GA_2f)$.
Then $WF(A_2f) \subset \{(p^{-1}(y),\xi) \mid (y,\xi) \in WF(f)\} \cup \{(q^{-1}(y),\xi) \mid (y,\xi) \in WF(f)\}$.
Therefore $WF(GA_2f) \subset \{((pq^{-1})^n(y),\xi) \mid n \in \mathbb{Z},\ n \geq 0,\ (y,\xi) \in WF(f)\}$. So if f has a singularity in (y_0,ξ_0), GA_2f can have a singularity in (y_0,ξ_0), too, and E might propagate these singularities to the left. However, we will show that they cancel along a strip through $(y_0,0,\lambda\xi_0,0)$, $\lambda > 0$, for $x < y_0$.

PROPOSITION 4.4.19. *Let* $h = 0$ *and* $(y_0,\xi_0) \in WF(f)$.
Assume $((qp^{-1})^n(y_0),\xi_0) \notin WF(f)$ *for every* $n \geq 1$.
Then $\{(y_0 - \frac{2}{3}(-t)^{\frac{3}{2}},t,\xi,-(-t)^{\frac{1}{2}}\xi) \mid t < 0, \xi\xi_0 > 0\} \cap WF(E_2(f,0)) = \emptyset$.

PROOF. $E_2(f,0) = E(f,-GA_2f)$ and $G = F_{11} + \Sigma_{n=1}^{\infty} (-F_{11}A_{12})^n F_{11}$. We have $((qp^{-1})^n(y_0),\xi_0) \notin WF(f)$ for $n \geq 1$, so it is sufficient to consider $E(f,-F_{11}A_2f)$.

As in Proposition 4.4.5, $A_2 \equiv A_{21} + A_{22}$ modulo a smoothing operator with

$$(A_{21}f)(x) = \iint dy d\xi e^{i(x-y+\frac{2}{3}\varphi^{\frac{3}{2}}(x))\xi} a_{21}(x,\xi)\chi_1(p(x),y)f(y)$$

for $f \in C_0^\infty$.

A_{21} is again a properly supported elliptic FIO and

$$a_{21}(x,\xi) = iAi'(0)\omega(\xi)\Big[e^{\pi i/3}H(\xi)a_+(-\varphi(x),\xi) + - e^{-\pi i/3}H(-\xi)a_-(-\varphi(x),\xi)\Big].$$

A_{22} is connected with the phase function $(x-y-\frac{2}{3}\varphi^{\frac{3}{2}}(x))\xi$.

This implies that we can restrict ourselves to the analysis of $E(f,-F_{11}A_{21}f)$.

A simple calculation gives:

$$(4.4.20) \qquad a_{21}(x,\xi) = \frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}(-\xi)^{\frac{2}{3}} a_{11}(x,\xi).$$

Define

$$f^* := \left[\frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}\chi(\xi)(-\xi)^{\frac{2}{3}}\hat{f}\right]^\vee$$

with χ smooth, zero for $|\xi| < 1$ and one for $|\xi| > 2$.

Compare this to formula (4.3.2)!

We claim that $A_{21}f - A_{11}f^*$ is smooth for $f \in \mathcal{E}'$.

A substitution of $z = p(x)$ turns A_{11} and A_{21} into ΨDOs.

$$A_{\ell 1}f(p^{-1}(z)) = \int d\xi e^{iz\xi}\sigma_\ell(z,\xi)\hat{f}(\xi), \quad \ell = 1,2, \ f \in \mathcal{S}.$$

Here

$$(4.4.21) \qquad \sigma_\ell(z,\xi) \sim a_{\ell 1}(p^{-1}(z),\xi), \quad \ell = 1,2.$$

See section 2.9, formula (2.9.2).

For $f \in C_0^\infty$ we have $f^* \in \mathcal{S}$ and

$$(A_{21}f - A_{11}f^*)(p^{-1}(z)) = \int d\xi e^{iz\xi}\sigma(z,\xi)\hat{f}(\xi),$$

with $\sigma(z,\xi) = \sigma_2(z,\xi) - \sigma_1(z,\xi)\frac{Ai'(0)}{Ai(0)}e^{-\pi i/3}\chi(\xi)(-\xi)^{\frac{2}{3}}$. Formulas (4.4.20) and (4.4.21) show that σ is a rapidly decreasing symbol. So indeed $A_{21}f - A_{11}f^*$ is smooth for $f \in \mathcal{E}'$. Since we can restrict ourselves to $f \in \mathcal{E}'$, we get

$$-F_{11}A_{21}f \equiv -F_{11}A_{11}f^* \equiv -f^* \quad (\text{modulo } C^\infty).$$

The analysis in section 4.3 now shows that modulo a smoothing operator,

$E(f,-f^*)$ is given by formula (4.3.4):

$$E(f,-f^*) \equiv -Ai'(0)\sqrt{3}\iint dyd\xi e^{i(x-y-\frac{2}{3}(-t)^{\frac{3}{2}})\xi}\ a(t,\xi)f(y).$$

This implies the proposition. □

REMARK 4.4.22. So we did not solve problem (4.4.1) exactly. But we reduced it to a similar problem with $f = 0$ and $h \in \mathcal{E}_0$. Presumably this problem is easier to handle, for instance in a numerical way. Alternatively, one might try to invert $I + T_2$. T_2 has a smooth kernel so possibly it is compact. Note that in order to keep T_1 and T_2 small (in some sense), R_1, R_2, A_{13} and A_{14} should be kept small. A_{14} can be dealt with by choosing an appropriate function ω (cf. section 2.8, example 3). The property of A_1 represented by Lemma 4.2.17 shows that for another choice of χ, χ_1 and χ_2 the second term in the definition of A_{13} vanishes and the first term can be added to A_{11}. Then we only have to deal with R_1, R_2. Unfortunately no further information is available about the possibility of choosing a parametrix F so that $FA_{11} - I$ is arbitrarily small (in some other sense than compactness). Therefore we leave it to this.

4.5. A mixed elliptic-hyperbolic problem.

In this section we consider a simple mixed elliptic-hyperbolic boundary value problem.

For $T \le \infty$ let Ω_T be the region given by

$$\Omega_T = \{(x,t) \mid (x < 0 \text{ and } 0 < t < T) \text{ or } (x \ge 0 \text{ and} -\left(\frac{3x}{2}\right)^{\frac{2}{3}} < t < T)\}.$$

Define Γ and Γ_+ as in section 4.3. Further

$$\Gamma_- := \{(x,t) \mid x < 0 \text{ and } t = 0\},$$

$$\Gamma_T := \{(x,t) \mid t = T\}.$$

As before, Γ and Γ_+ can be identified with $\mathbb{R}^+$, Γ_- with $\mathbb{R}^-$ and Γ_T with $\mathbb{R}$. Note that $\text{bnd}\ \Omega = \text{cl}(\Gamma_- \cup \Gamma \cup \Gamma_T)$.

For $T < \infty$ we consider the problem

$$(4.5.1)\qquad \begin{cases} \mathcal{T}u = 0 \ \text{ in } \Omega_T, \\ u|_{\Gamma_-} = f^-, \\ u|_{\Gamma} = h, \\ u|_{\Gamma_T} = k. \end{cases}$$

We will assume $f^- \in \mathcal{E}'(\Gamma_-)$, $h \in \mathcal{E}'(\Gamma)$ and $k \in \mathcal{E}'(\Gamma_T)$.

Let Ω^- be the open part of Ω_T bounded by Γ_+ and Γ (the hyperbolic part).
Let Ω_T^+ be the open part of Ω_T bounded by Γ_-, Γ_+ and Γ_T (the elliptic part).

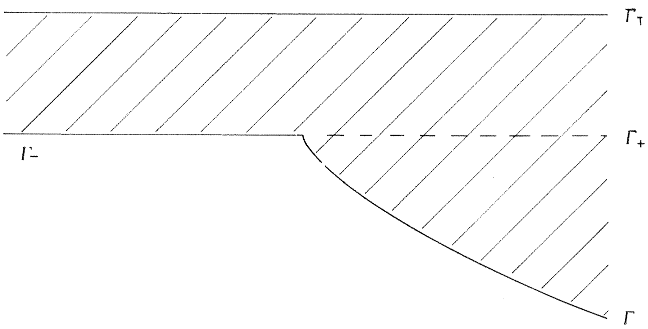

Fig. 18: Ω_T.

Without the boundary condition on $t = T$ (that is: $T = \infty$) but with the condition u is bounded for $|(x,t)| \to \infty$, $t \geq 0$, a solution for this problem is wellknown in the case f^- and h continuous and integrable (see for instance Von Wolfersdorf [28]). The solution is then obtained as follows. Determine f^+ on Γ_+ and solutions u_- and u_+ in Ω^- and Ω_∞^+ so that $u_-|_{\Gamma_+} = f^+$, $u_-|_\Gamma = h$, $u_+|_{\Gamma_- \cup \Gamma_+} = f^- + f^+$ and $\frac{\partial}{\partial t}(u_+ - u_-)|_{\Gamma_+} = 0$. Then f^+ can be found by solving an integral equation on a halfspace. However, this method doesnot make it clear that one obtains a solution in Ω and not only in $\Omega^- \cup \Omega_\infty^+$.
We will choose a similar approach and allow distributional data which are nice at (0,0). Actually we do not arrive at an integral equation for the determination of f^+ but rather on (one of) the proof(s) of solvability of such an equation.

From now on we take $T = 1$ and omit the subscript T. For other values the problem can be treated in the same way.
The problem in Ω^- we discussed in section 4.3. Here we discuss the problem in Ω^+. From equation (4.2.3) it is clear that a formal solution for the problem

$$(4.5.2) \qquad \begin{cases} Tu = 0 & \text{in } \Omega^+, \\ u|_{t=0} = f, \\ u|_{t=1} = k \end{cases}$$

is given by

$$\frac{1}{2\pi}\int e^{ix\xi}(c_1(\xi)Ai(t|\xi|^{\frac{2}{3}})+c_2(\xi)Bi(t|\xi|^{\frac{2}{3}}))d\xi,$$

with

$$\begin{pmatrix}c_1\\c_2\end{pmatrix} = \frac{1}{Ai(0)Bi(|\xi|^{\frac{2}{3}}) - Bi(0)Ai(|\xi|^{\frac{2}{3}})}\begin{pmatrix}Bi(|\xi|^{\frac{2}{3}}) & -Bi(0)\\ -Ai(|\xi|^{\frac{2}{3}}) & Ai(0)\end{pmatrix}\begin{pmatrix}\hat{f}\\ \hat{k}\end{pmatrix}.$$

Then, again formally, the derivative g of this expression with respect to t at t = 0 is given by

$$(4.5.3)\qquad \hat{g}(\xi) = c_3(\xi)\hat{f}(\xi) + c_4(\xi)\hat{k}(\xi),$$

with

$$\begin{pmatrix}c_3\\c_4\end{pmatrix} = \frac{|\xi|^{\frac{2}{3}}}{Ai(0)Bi(|\xi|^{\frac{2}{3}}) - Bi(0)Ai(|\xi|^{\frac{2}{3}})} \times \begin{pmatrix}Ai'(0)Bi(|\xi|^{\frac{2}{3}}) - Bi'(0)Ai(|\xi|^{\frac{2}{3}})\\ \frac{1}{\pi}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{Ai'(0)}{Ai(0)}|\xi|^{\frac{2}{3}}\\ 0\end{pmatrix} + \frac{|\xi|^{\frac{2}{3}}}{Ai(0)Bi(|\xi|^{\frac{2}{3}})-Bi(0)Ai(|\xi|^{\frac{2}{3}})}\begin{pmatrix}\frac{-1}{\pi Ai(0)}Ai(|\xi|^{\frac{2}{3}})\\ \frac{1}{\pi}\end{pmatrix}.$$

LEMMA 4.5.4. c_3 *and* c_4 *are real analytic. For* ξ *real we have*

$$c_3(\xi) = \frac{Ai'(0)}{Ai(0)}|\xi|^{\frac{2}{3}} + e^{-\frac{4}{3}|\xi|}\sigma_1(\xi),$$

$$c_4(\xi) = e^{-\frac{2}{3}|\xi|}\sigma_2(\xi),$$

with σ_1 *satisfying* $S^{\frac{2}{3}}_{1,0}$-, σ_2 *satisfying* $S^{\frac{5}{6}}_{1,0}$*-estimates, both for large* ξ.

PROOF.
$$Ai'(0)Bi(|\xi|^{\frac{2}{3}})-Bi'(0)Ai(|\xi|^{\frac{2}{3}}) = -\frac{1}{\pi}y_1(|\xi|^{\frac{2}{3}}) = -\frac{1}{\pi}\sum_{n=0}^{\infty}\alpha_n\xi^2,$$

$$|\xi|^{-\frac{2}{3}}\left[Ai(0)Bi(|\xi|^{\frac{2}{3}})-Bi(0)Ai(|\xi|^{\frac{2}{3}})\right] = \frac{1}{\pi}|\xi|^{-\frac{2}{3}}y_2(|\xi|^{\frac{2}{3}}) = \frac{1}{\pi}\sum_{n=0}^{\infty}\beta_n\xi^2.$$

For the definition of y_1 and y_2, see section A.2.
So these expressions are analytic and c_3 and c_4 are real analytic in $\xi = 0$.
For $z > 0$:

$$Ai(0)Bi(z) - Bi(0)Ai(z) = 2Ai(0)(\frac{z}{3})^{\frac{1}{2}}e^{-\pi i/6}J_{\frac{1}{3}}(\frac{2}{3}iz^{\frac{3}{2}}).$$

See formula A.2.6. Now $J_{\frac{1}{3}}(w)$ has only real zeros, so c_3 and c_4 are real

analytic everywhere.
Now $Ai(|\xi|^{\frac{2}{3}}) = \exp(-\frac{2}{3}|\xi|)ai_1(1,\xi)$, $ai_1(1,\xi)$ a symbol in $S^{-\frac{1}{6}}_{1,0}$ for ξ large,
and $Bi(|\xi|^{\frac{2}{3}}) = \exp(\frac{2}{3}|\xi|)b(\xi)$, $b(\xi)$ an elliptic symbol in $S^{-\frac{1}{6}}_{1,0}$ for ξ large.
See section A.5.
The ellipticity of b then shows that

$$e^{\frac{2}{3}|\xi|}c_4(\xi) = \frac{\frac{1}{\pi}|\xi|^{\frac{2}{3}}}{Ai(0)b(\xi)-Bi(0)ai(1,\xi)e^{-\frac{4}{3}|\xi|}} \in S^{\frac{5}{6}}_{1,0} \text{ for large } \xi,$$

$$e^{\frac{4}{3}|\xi|}\left(c_3(\xi) - \frac{Ai'(0)}{Ai(0)}|\xi|^{\frac{2}{3}}\right) =$$

$$= -\frac{1}{\pi Ai(0)} \frac{|\xi|^{\frac{2}{3}}ai(1,\xi)}{Ai(0)b(\xi) - Bi(0)ai(1,\xi)e^{-\frac{4}{3}|\xi|}} \in S^{\frac{2}{3}}_{1,0} \text{ for large } \xi. \quad \square$$

Lemma 4.5.4 shows that the multiplications in equation (4.5.3) are welldefined if f and k belong to $S'(\mathbb{R})$. Further g satisfies condition (4.2.11) for such f and k. From Proposition 4.2.13 it then follows that with g given by equation (4.5.3) $E^1(f,g)$ is a solution for problem (4.5.2).

We continue by determining $f^+ \in M_\alpha \cap S'$, $\alpha > \frac{5}{6}$, so that for $f = f^+ + f^-$ the distributions g^+, defined by equation (4.5.3), and g^-, defined by equations (4.3.2) and (4.3.10), are equal on Γ_+.
The solutions $E(f^+,g^-)$ in Ω^- and $E^1(f,g^+)$ in Ω^+ then satisfy the same boundary conditions on Γ_+. Remark 4.2.15 says that $E^1(f,g^+) = E(f,g^+)$ for $t < 0$. The properties of U_t and V_t show that $E(f,g^+) = E(f^+,g^+) = E(f^+,g^-)$ in Ω^-. Now $E^1(f,g^+)$ is a solution of $Tu = 0$ for $t \leq 1$. Therefore we get

PROPOSITION 4.5.5. *The construction explained above provides a solution for problem* (4.5.1) *in* Ω *and not only in* $\Omega^+ \cup \Omega^-$. $\square$

We proceed to determine f^+. The equation for f^+ is

(4.5.6) $$\left[c_3\hat{f}^+ + c_3\hat{f}^- + c_4\hat{k}\right]^\vee = -\frac{1}{\Gamma(-\frac{2}{3})}\frac{Ai'(0)}{Ai(0)}\, x_+^{-\frac{5}{3}} * f^+ + Qh \quad \text{on } \Gamma_+.$$

Here Q is defined by equation (4.3.10).
First we will show how a possible candidate for f^+ can be obtained by means of a derivation in which some steps may be hard to justify. Afterwards we will show that this candidate is indeed a solution.

Rewrite equation (4.5.6) as follows:

(4.5.7) $$Ff^+ = \Psi \quad \text{on } \Gamma_+$$

with

$$Ff^+ := \left[c_3\hat{f}^+\right]^\vee + \frac{1}{\Gamma(-\frac{2}{3})}\frac{Ai'(0)}{Ai(0)}\, x_+^{-\frac{5}{3}} * f^+, \quad \Psi = Qh - [c_3\hat{f}^- + c_4\hat{k}]^\vee.$$

Recall that $\frac{1}{\Gamma(-\frac{2}{3})}\, x_+^{-\frac{5}{3}} = \left[e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}}\right]^\vee$.

For $\xi \neq 0$ we have

$$c_3(\xi) + \frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}} = \frac{Ai'(0)}{Ai(0)}\left(|\xi|^{\frac{2}{3}} + e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}}\right) + $$
$$+ e^{-\frac{4}{3}|\xi|}\sigma_1(\xi).$$

Let us omit the last term for the moment (then we neglect a smoothing operator!). We have

$$|\xi|^{\frac{2}{3}} + e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}} = (-\xi+i0)^{\frac{2}{3}}(e^{-\pi i/3} + e^{-2\pi i/3}H(\xi) + H(-\xi))$$
$$= \tfrac{3}{2}e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}}\left(1 + \frac{1}{i\sqrt{3}}\,\mathrm{sign}\;\xi\right).$$

With $u := \left[\frac{3}{2}e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}}\right]^\vee * f^+$ we arrive at the integral equation

$$\frac{Ai(0)}{Ai'(0)}\,\Psi = u + \frac{1}{i\sqrt{3}}\frac{1}{2\pi}\int e^{ix\xi}\,\mathrm{sign}\,(\xi)\hat{u}d\xi = u + \frac{1}{\pi\sqrt{3}}\left(\frac{1}{x} * u\right), \quad x > 0.$$

Note that $\mathrm{supp}(u) \subset \overline{\mathbb{R}^+} \Leftrightarrow \mathrm{supp}(f^+) \subset \overline{\mathbb{R}^+}$.

Similar integral equations are encountered in the work of other authors discussing problem (4.5.1). For Ψ, u in $L_2[0,\infty]$ a solution is given by

$$u = \frac{3}{4}\frac{Ai(0)}{Ai'(0)}\left[\Psi - \frac{1}{\pi\sqrt{3}}\, x_+^{-\frac{1}{6}}\int_0^\infty \frac{\Psi(y)y^{\frac{1}{6}}}{x-y}dy\right].$$

See Hochstadt [14], page 190.

However, we do not think this formula to be suitable as a basis for a generalization to our situation. But the method of constructing this solution suggests how to get another solution formula. For we also have

$$|\xi|^{\frac{2}{3}} + e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}} = e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}}(1 + e^{-\pi i/3}H(\xi) + e^{\pi i/3}H(-\xi))$$
$$= \sqrt{3}\, e^{-\pi i/3}\,(-\xi+i0)^{\frac{1}{2}}(\xi+i0)^{\frac{1}{6}}.$$

With $v := \left[\sqrt{3}\, e^{-\pi i/3}\,(-\xi+i0)^{\frac{1}{2}}\right]^\vee * f^+$ we get the equation

$$\frac{Ai(0)}{Ai'(0)}\,\Psi = \left[(\xi+i0)^{\frac{1}{6}}\right]^\vee * v \quad \text{for } x > 0.$$

Because $\mathrm{supp}\left[(\xi+i0)^{\frac{1}{6}}\right]^\vee \subset \overline{\mathbb{R}^-}$ then it follows that we must have

$$v = \frac{Ai(0)}{Ai'(0)}\left[(\xi+i0)^{-\frac{1}{6}}\right]^\vee * \Psi \quad \text{for } x > 0.$$

Therefore a possible solution is

$$f^+ := \frac{e^{\pi i/3}}{\sqrt{3}} \frac{Ai(0)}{Ai'(0)} \left[(-\xi+i0)^{-\frac{1}{2}}\right]^\vee * H(x)\left(\left[(\xi+i0)^{-\frac{1}{6}}\right]^\vee * \Psi\right).$$

After this introduction we return to equation (4.5.7).
Note that we factorized $|\xi|^{\frac{2}{3}} + e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}}$ in two factors being analytically continuable for $\mathrm{Im}\,\xi < 0$ or $\mathrm{Im}\,\xi > 0$ so that they are Fourier transforms of distributions with support in $\overline{\mathbb{R}^+}$ and $\overline{\mathbb{R}^-}$ respectively. We will factorize $c_3(\xi) + \frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}}$ in a similar way.
This is a continuous function which behaves for $|\xi| \to 0$ like $-1 + \frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}}$ because $c_3(0) = -1$ and for $|\xi| \to \infty$ like

$$\frac{Ai'(0)}{Ai(0)}\left(|\xi|^{\frac{2}{3}} + e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}}\right) =$$

$$= \sqrt{3}\, \frac{Ai'(0)}{Ai(0)}|\xi|^{\frac{2}{3}}\left(H(\xi)e^{\pi i/6} + H(-\xi)e^{-\pi i/6}\right).$$

<u>LEMMA 4.5.8</u>. $\dfrac{c_3(\xi) + \frac{Ai'(0)}{Ai(0)} e^{-\pi i/3} (-\xi+i0)^{\frac{2}{3}}}{-1 + \frac{Ai'(0)}{Ai(0)} e^{-\pi i/3} (-\xi+i0)^{\frac{2}{3}}}$ *is in* $C^1(\mathbb{R})$.

<u>PROOF</u>. $\quad -1 + \frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}} = -1 + \frac{Ai'(0)}{Ai(0)} e^{\pm\pi i/3}|\xi|^{\frac{2}{3}}$ for $\xi \gtrless 0$.
$\frac{Ai'(0)}{Ai(0)} < 0$ so the real part is ≤ -1. It is therefore clear that the expression is smooth for $\xi \neq 0$ and continuous for all ξ. Now $c_3(\xi)$ is real analytic so $c_3(\xi) = -1+\xi r(\xi)$, r smooth, for ξ small. But then the expression is equal to

$$1 + \frac{\xi r(\xi)}{-1 + \frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}}}.$$

For $\xi \neq 0$ this is differentiable and the derivative has a continuous extension for $\xi = 0$. □

Note that the quotient behaves like $\sqrt{3}\, e^{\mp\pi i/6}$ for $\xi \to \pm\infty$.
We define the function $s(\xi)$ by

$$(4.5.9) \qquad s(\xi) := \frac{c_3(\xi) + \frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}}}{\sqrt{3}(\xi+i)^{\frac{1}{6}}(-\xi+i)^{-\frac{1}{6}}(-1 + \frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}})}.$$

Then s is C^1 and $s \sim 1$ for $|\xi| \to \infty$.

<u>LEMMA 4.5.10</u>. $-\frac{Ai'(0)}{Ai(0)} e^{\pm\pi i/3}|\xi|^{\frac{2}{3}}(1-s(\xi)) \to 1$ for $\xi \to \pm\infty$.

PROOF. $1 - s(\xi) =$

$$= \frac{-1 + \frac{Ai'(0)}{Ai(0)} e^{\pm\pi i/3} |\xi|^{\frac{2}{3}} \left(1 - e^{\mp\pi i/6} \frac{(-\xi+i)^{\frac{1}{6}}}{(\xi+i)^{\frac{1}{6}}}\right)}{-1 + \frac{Ai'(0)}{Ai(0)} e^{\pm\pi i/3} |\xi|^{\frac{2}{3}}} + \text{ a rapidly decreasing function.}$$

Now

$$1 - e^{\mp\pi i/6} \frac{(-\xi+i)^{\frac{1}{6}}}{(\xi+i)^{\frac{1}{6}}} = 1 - e^{\mp\pi i/6} \frac{(\mp 1 \pm \frac{i}{\xi})^{\frac{1}{6}}}{(\pm 1 \pm \frac{i}{\xi})^{\frac{1}{6}}} = 1 - \frac{(1 - \frac{i}{\xi})^{\frac{1}{6}}}{(1 + \frac{i}{\xi})^{\frac{1}{6}}} \sim \frac{i}{3(\xi+i)}.$$

Using this the result follows. □

It is clear that $|\arg s(\xi)| < \pi$, $s(\xi) \neq 0$ and s bounded. With Lemma 4.5.10, $\varepsilon < |s(\xi)| < M$ for some $0 < \varepsilon < M$. So with log z the principal branch of the logarithm, $\log s(\xi)$ is welldefined.

LEMMA 4.5.11. $s \in S^0_{1,0}(\mathbb{R} \setminus 0)$, *that is:* s *satisfies* $S^0_{1,0}$*-estimates for* $|\xi| \geq \delta > 0$, δ *arbitrary.*

PROOF. $(-\xi+i)^{\frac{1}{6}}\left[c_3(\xi) + \frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}}\right]$ belongs to $S^{\frac{5}{6}}_{1,0}$ for ξ large.

$(\xi+i)^{\frac{1}{6}}\left[-1 + \frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}}\right]$ belongs to $S^{\frac{5}{6}}_{1,0}$ for ξ large

and is elliptic for $\xi \neq 0$. So $s \in S^0_{1,0}$ for ξ large. □

COROLLARY 4.5.12.

1. $\log s(\xi) \in C^1(\mathbb{R}) \cap L_2(\mathbb{R})$ *and* $\frac{d}{d\xi}\log s(\xi)$ *is bounded.*
2. $\log s(\xi)$ *satisfies* $S^0_{1,0}$*-estimates for* $|\xi| \geq \delta > 0$.

PROOF. 1. $\log s(\xi) \in C^1(\mathbb{R})$ is evident. For $|\xi| \to \infty$: $\log s(\xi) = \log(1 - (1-s(\xi))) \sim -(1-s(\xi))$. So Lemma 4.5.10 shows that $\log s(\xi) \in L_2(\mathbb{R})$. $\frac{d}{d\xi}\log s(\xi) = \frac{s'(\xi)}{s(\xi)}$, which is clearly bounded.
2. $s'(\xi) \in S^{-1}_{1,0}$, $s(\xi) \in S^0_{1,0}$ and s is elliptic for large ξ. So $\frac{d}{d\xi}\log s(\xi) \in S^{-1}_{1,0}$. Since $\log s(\xi)$ is bounded, the result follows. □

PROPOSITION 4.5.13. ∃ *function* u(z) *analytic for* Im z ≠ 0 *and bounded for* $|\text{Im } z| \geq \delta$, $\delta > 0$ *arbitrary, so that* $\frac{1}{u(z)}$ *has the same properties as* u(z) *and*
∃ *bounded continuous functions* $s^+(\xi)$ *and* $s^-(\xi)$ *so that* $\frac{1}{s^+}$ *and* $\frac{1}{s^-}$ *are bounded and continuous also and with* $u^{\pm}_{\varepsilon}(\xi) := u(\xi \pm i\varepsilon)$:
1. $s(\xi) = \frac{s^+(\xi)}{s^-(\xi)}$.

2. $\sup |u_\varepsilon^+(\xi) - s^+(\xi)| \to 0$ *for* $\varepsilon \downarrow 0$.

3. $\sup |u_\varepsilon^-(\xi) - s^-(\xi)| \to 0$ *for* $\varepsilon \downarrow 0$.

4. $\sup \left|\frac{1}{u_\varepsilon^+(\xi)} - \frac{1}{s^+(\xi)}\right| \to 0$ *for* $\varepsilon \downarrow 0$.

5. $\sup \left|\frac{1}{u_\varepsilon^-(\xi)} - \frac{1}{s^-(\xi)}\right| \to 0$ *for* $\varepsilon \downarrow 0$.

PROOF. Define

$$u(\xi+i\varepsilon) := \exp\left[-\frac{1}{2\pi i}\frac{1}{\xi+i\varepsilon} * \log s(\xi)\right], \quad \varepsilon \neq 0,$$

$$s^+(\xi) := \exp\left[\tfrac{1}{2}\log s(\xi) - \frac{1}{2\pi i}\mathrm{vp}\frac{1}{\xi} * \log s(\xi)\right],$$

$$s^-(\xi) := \exp\left[-\tfrac{1}{2}\log s(\xi) - \frac{1}{2\pi i}\mathrm{vp}\frac{1}{\xi} * \log s(\xi)\right].$$

Corollary 4.5.12 shows that $\log s(\xi) \in C^1 \cap L_2$ and $\frac{d}{d\xi}\log s(\xi)$ is bounded. So Lemma A.6.4 shows that $\frac{1}{\xi+i\varepsilon} * \log s(\xi)$ is analytic in $z = \xi+i\varepsilon$ for $\varepsilon \neq 0$ and bounded for $|\varepsilon| \geq \delta$. Further Lemma A.6.1 shows that $\mathrm{vp}\frac{1}{\xi} * \log s(\xi)$ is bounded and continuous. Then it is clear that $u(\xi+i\varepsilon)$ and $\frac{1}{u(\xi+i\varepsilon)}$ are analytic in $\xi+i\varepsilon$ for $\varepsilon \neq 0$ and bounded for $|\varepsilon| \geq \delta$. Also s^+, s^-, $\frac{1}{s^+}$ and $\frac{1}{s^-}$ are bounded and continuous. That $s(\xi) = s^+(\xi)/s^-(\xi)$ is evident.
The boundedness of s^+, s^-, $\frac{1}{s^+}$ and $\frac{1}{s^-}$ together with Lemma A.6.5 then shows that the other statements are true. See also Conway [5], page 162, Lemma 5.7. □

REMARK 4.5.14. This Proposition and its proof are a modification of Theorems 7 and 8 in Hochstadt [14], page 191 and 192. The main difference is that we took care of uniform convergence.

Let us call $c_3(\xi) + \frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}} =: q(\xi)$.
We are ready now to factorize $q(\xi)$.

Define

$$q^+(\xi) := \sqrt{3}(\xi+i)^{\frac{1}{6}} s^+(\xi),$$

$$q^-(\xi) := (-\xi+i)^{-\frac{1}{6}}\left(-1 + \frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}(-\xi+i0)^{\frac{2}{3}}\right)\frac{1}{s^-(\xi)}.$$

Further

$$q^+(z) := \sqrt{3}(z+i)^{\frac{1}{6}} u(z) \quad \text{for } \operatorname{Im} z > 0,$$

$$q^-(z) := (-z+i)^{-\frac{1}{6}}\left(-1 + \frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}(-z)^{\frac{2}{3}}\right)\frac{1}{u(z)} \quad \text{for } \operatorname{Im} z < 0.$$

PROPOSITION 4.5.15.

1. $q^+(\xi)$, $q^-(\xi)$, $\frac{1}{q^+(\xi)}$ *and* $\frac{1}{q^-(\xi)}$ *are continuous functions which belong to* $S'(\mathbb{R}_\xi)$, *so that* $q(\xi) = q^-(\xi)q^+(\xi)$.
2. $q^+(z)$ *and* $\frac{1}{q^+(z)}$ *are analytic for* $\operatorname{Im} z > 0$, $q^-(z)$ *and* $\frac{1}{q^-(z)}$ *for* $\operatorname{Im} z < 0$.
3. *If* $q^\pm_\varepsilon(\xi) := q^\pm(\xi \pm i\varepsilon)$, $\varepsilon > 0$, *then* $q^\pm_\varepsilon(\xi)$ *and* $\frac{1}{q^\pm_\varepsilon(\xi)}$ *belong to* $O_M(\mathbb{R}_\xi)$ $\forall\, \varepsilon > 0$ *and* $q^\pm_\varepsilon(\xi) \to q^\pm(\xi)$, $\frac{1}{q^\pm_\varepsilon(\xi)} \to \frac{1}{q^\pm(\xi)}$ *in* S' $(\varepsilon \downarrow 0)$.
4. $q^\pm(\xi)$ *and* $\frac{1}{q^\pm(\xi)}$ *are Fourier transforms of distributions in* S' *with support contained in* $\overline{\mathbb{R}^\mp}$.

PROOF. The first and second part follow easily from Proposition 4.5.13 and formula (4.5.9).
Clearly $q^\pm_\varepsilon(\xi)$ and $\frac{1}{q^\pm_\varepsilon(\xi)}$ are smooth for $\varepsilon > 0$. In order to show that they belong to O_M, it is sufficient to show that $u^\pm_\varepsilon$ and $\frac{1}{u^\pm_\varepsilon}$ (see Proposition 4.5.13) do. These functions are bounded. Furthermore

$$\frac{d^n}{d\xi^n}\left[\frac{1}{\xi+i\varepsilon} * \log s(\xi)\right] = (-1)^n n!\,\frac{1}{(\xi+i\varepsilon)^{n+1}} * \log s(\xi) \text{ is bounded.}$$

Then the derivatives of the functions $u^\pm_\varepsilon$ and $\frac{1}{u^\pm_\varepsilon}$ are bounded, too. So they belong to O_M. The convergence in S' is clear, for if we divide any of the functions $q^\pm_\varepsilon(\xi)$ and $\frac{1}{q^\pm_\varepsilon(\xi)}$ by $1/(1+\xi^2)$, they are bounded uniformly for $|\varepsilon| \le 1$ by an integrable function. Here we use the uniform convergence mentioned in Remark 4.5.14. Finally let $\varphi \in C_0^\infty(\mathbb{R})$ have support contained in $\overline{\mathbb{R}^+}$.

$$\langle (q^+)^\vee, \varphi\rangle = \langle q^+, \check{\varphi}\rangle = \lim_{\varepsilon\downarrow 0} \langle q^+_\varepsilon, \check{\varphi}\rangle = \lim_{\varepsilon\downarrow 0} \int q^+(\xi+i\varepsilon)\check{\varphi}(\xi)d\xi.$$

Now $(1 - (\partial^2/\partial x^2))\varphi$ has support in $\overline{\mathbb{R}^+}$ as well so $(1+\xi^2)\check{\varphi}(\xi)$ is an analytic function which is bounded for $\operatorname{Im} \xi \ge 0$. So with contour integration:

$$\int q^+(\xi+i\varepsilon)\check{\varphi}(\xi)d\xi = 0.$$

Then it follows that $\langle (q^+)^\vee, \varphi\rangle = 0$. So $(q^+)^\vee$ has support contained in $\overline{\mathbb{R}^-}$. The supports of the other distributions can be determined in a similar way.

□

REMARK 4.5.16. Of course, the statement about the supports can be generalized to distributions having analytic continuations that give continuous approximations in S'.

For future use we state one more lemma.

LEMMA 4.5.17. q^+, q^-, $\frac{1}{q^+}$ *and* $\frac{1}{q^-}$ *can be written as the sum of a continuous function with compact support and an elliptic symbol in* $S^m_{\frac{1}{2}\rho,0}$ *for some* m, $0 < \rho < \frac{1}{2}$.

PROOF. For ξ large, Lemma 4.5.11 shows that s satisfies $S^0_{1,0}$-estimates. Because $\lim_{|\xi|\to\infty} s(\xi) = 1$, the same holds for $|s(\xi)|^{\pm\frac{1}{2}}$ (see section 2.8, paragraph (2.8.3)). According to Corollary 4.5.12, so does $\log s(\xi)$.
Let ψ be a C_0^∞-function, equal to 1 in a neighbourhood of 0. Then $\mathrm{vp}\frac{1}{\xi} * \psi(\xi)\log s(\xi)$ is the sum of a C_0^0-function and a symbol in $S^{-1}_{1,0}$ (see Lemma A.6.2). Lemma A.6.6 and Corollary 4.5.12 show that

$$\frac{d}{d\xi}\left[\mathrm{vp}\frac{1}{\xi} * (1-\psi)\log s(\xi)\right] = \mathrm{vp}\frac{1}{\xi} * \frac{d}{d\xi}(1-\psi)\log s.$$

Here $\frac{d}{d\xi}(1-\psi)\log s \in S^{-1}_{\rho,0}$, $0 < \rho \le 1$, so according to Lemma A.6.3:

$$\frac{d}{d\xi}\left[\mathrm{vp}\frac{1}{\xi} * (1-\psi)\log s(\xi)\right] \in S^m_{\frac{1}{2}\rho,0} \text{ for } m > -\tfrac{1}{2}+\tfrac{\rho}{2},\ 0 < \rho \le 1.$$

But then $\mathrm{vp}\frac{1}{\xi} * (1-\psi)\log s \in S^0_{\frac{1}{2}\rho,0}$ if $0 < \rho < \frac{1}{2}$ because $\mathrm{vp}\frac{1}{\xi} * (1-\psi)\log s$ is a bounded function (see Lemma A.6.1).
We conclude that $\mathrm{vp}\frac{1}{\xi} * \log s = v_1 + \sigma$ with $v_1 \in C_0^0$, $\sigma \in S^0_{\frac{1}{2}\rho,0}$, $0 < \rho < \frac{1}{2}$. Then $e^{\pm(1/2\pi i)\sigma} \in S^0_{\frac{1}{2}\rho,0}$ as well and $e^{\pm(1/2\pi i)v_1} = 1$ for large ξ.
The rest of the proof is now straightforward. □

We now return again to equation (4.5.7).
Purely formally we deduce:

$$Ff^+ = [q(\xi)\hat{f}^+(\xi)]^\vee = [q^+(\xi)q^-(\xi)\hat{f}^+(\xi)]^\vee = \Psi \quad \text{for } x > 0.$$

So

$$[q^-(\xi)\hat{f}^+(\xi)]^\vee = \left[\frac{1}{q^+(\xi)}\right]^\vee * \Psi \quad \text{for } x > 0,$$

$$[q^-(\xi)\hat{f}^+(\xi)]^\vee = H(x)\left(\left[\frac{1}{q^+}\right]^\vee * \Psi\right)$$

and we get

$$f^+(x) = \left[\frac{1}{q^-}\right]^\vee * H(x)\left(\left[\frac{1}{q^+}\right]^\vee * \Psi\right).$$

The main problem we encounter when we try to show that this formal solution indeed solves equation (4.5.7) is that convolution between arbitrary distributions is not necessarily welldefined. If it is, it is not necessarily associative. Therefore we will only give a partial solution for problem (4.5.1). That is, we give a solution u that satisfies $u|_\Gamma = h$ only for

$x < R$. Here $R > 0$ can be chosen arbitrarily large.
Let ψ_R, $R > 1$ be a C_0^∞-function so that

$$\psi_R(x) = \begin{cases} 1 & \text{for } \frac{1}{R} < x < R \\ 0 & \text{for } x < \frac{1}{2R},\ x > R+1 \end{cases} .$$

Define $\Psi_R := \psi_R Qh - [c_3\hat{f}^- + c_4\hat{k}]^\vee$.
We will consider the equation

(4.5.18) $\quad Ff^+ = \Psi_R$ on Γ_+, $f^+ \in M_\alpha \cap S'$ for some $\alpha > \frac{5}{6}$.

We claim that a solution is given by

(4.5.19) $\quad f^+ = \left[\frac{1}{q^-}\right]^\vee * H(x)\left(\left[\frac{1}{q^+}\right]^\vee * \Psi_R\right)$.

LEMMA 4.5.20. $H(x)\left(\left[\frac{1}{q^+}\right]^\vee * \Psi_R\right)$ *is a welldefined element of* $S'(\mathbb{R})$. *Its singularities for* $x > 0$ *are those of* $\psi_R Qh$.

PROOF. $\left[\frac{1}{q^+}\right]^\vee \in S'$ and $\Psi_R \in O_C'$ because of Lemma 4.5.4. So the convolution is welldefined. From Lemma 4.5.17 we know that $\frac{1}{q^+}$ is the sum of a continuous function with compact support and an elliptic symbol. Then $\left[\frac{1}{q^+}\right]^\vee * \psi_R Qh$ is an elliptic ΨDO modulo a smoothing operator. Lemma 4.5.4 shows that $\left[\frac{1}{q^+}\right]^\vee * (c_4\hat{k})^\vee$ is smooth and that $\left[\frac{1}{q^+}\right]^\vee * [c_3\hat{f}^-]^\vee$ is an elliptic ΨDO modulo a smoothing operator. Therefore $\left[\frac{1}{q^+}\right]^\vee * \Psi_R$ is smooth in a neighbourhood of $x = 0$, for f^- and Qh are. Multiplication with $H(x)$ is then welldefined and the singularities of $H(x)\left(\left[\frac{1}{q^+}\right]^\vee * \Psi_R\right)$ are those of $\psi_R Qh$. □

LEMMA 4.5.21. f^+ *defined by formula* (4.5.19) *is a welldefined element of* $S'(\mathbb{R})$. $f^+ \in M_\alpha$ *for* $\alpha < 1$.
The singularities of f^+ *for* $x > 0$ *are those of* $\psi_R Qh$.

PROOF. f^+, being the convolution of two distributions with support in $\overline{\mathbb{R}^+}$, is welldefined and $\text{supp}(f^+) \subset \overline{\mathbb{R}^+}$.
Lemma A.7.3 gives $f^+ \in S'(\mathbb{R})$.
In order to show that $f^+ \in M_\alpha$ for $\alpha < 1$ it is sufficient to analyze $\left[\frac{1}{q^-}\right]^\vee * H(x)\varphi(x)$ with $\varphi \in C_0^\infty$.
Because $|\xi\widehat{H\varphi}|$ is bounded, $|\xi|^{\frac{3}{2}}\frac{1}{q^-}\widehat{H\varphi}$ is bounded as well. So $\left[\frac{1}{q^-}\right]^\vee * H\varphi \in H^\alpha(\mathbb{R})$ for $\alpha < 1$. Then it easily follows $f^+ \in M_\alpha$, $\alpha < 1$, because $u \in H^\alpha \Rightarrow \psi u \in H^\alpha$ if $\psi \in C_0^\infty$.
For the determination of the singularities of f^+ we can assume $H(x)\left(\left[\frac{1}{q^+}\right]^\vee * \Psi_R\right) \in E'$. But then the proof is similar to the proof

in Lemma 4.5.20. □

LEMMA 4.5.22. $H(x)([\frac{1}{q_\varepsilon^+}]^\vee * \Psi_R)$ *is a welldefined element of* $\mathcal{O}_C'$. *Its singularities for* $x > 0$ *are those of* $\psi_R Qh$.

$$H(x)\left(\left[\frac{1}{q_\varepsilon^+}\right]^\vee * \Psi_R\right) \to H(x)\left(\left[\frac{1}{q^+}\right] * \Psi_R\right) \textit{ in } S' \textit{ for } \varepsilon \downarrow 0.$$

PROOF. $\frac{1}{q_\varepsilon^+}$ is an elliptic symbol. This can be proved by methods similar to the ones used in the proof of Lemma 4.5.17. As in Lemma 4.5.20, $[\frac{1}{q_\varepsilon^+}]^\vee * \Psi_R$ is smooth in $x = 0$ and its singularities for $x > 0$ are those of $\psi_R Qh$. It is clearly an element of $\mathcal{O}_C'$. Choose $\delta > 0$ so that f^- and Qh are smooth for $|x| < \delta$ and $\chi \in C_0^\infty$ so that

$$\chi(x) = \begin{cases} 1 & \text{for } |x| < \frac{1}{2}\delta \\ 0 & \phantom{\text{for }} |x| \geq \frac{3}{4}\delta \end{cases}.$$

Then $\varphi_\varepsilon := \chi([\frac{1}{q_\varepsilon^+}]^\vee * \Psi_R)$ and $\varphi_0 := \chi([\frac{1}{q^+}]^\vee * \Psi_R)$ are smooth. $H(x)\varphi_\varepsilon(x) \in E' \subset \mathcal{O}_C'$ and that $H(x)(1-\chi)([\frac{1}{q_\varepsilon^+}]^\vee * \Psi_R) \in \mathcal{O}_C'$ is clear. Finally $[\frac{1}{q_\varepsilon^+}]^\vee \to [\frac{1}{q^+}]^\vee$ in S', $\Psi_R \in \mathcal{O}_C'$ so $[\frac{1}{q_\varepsilon^+}]^\vee * \Psi_R \to [\frac{1}{q^+}]^\vee * \Psi_R$ in S'. Then $H\varphi_\varepsilon \to H\varphi_0$ in S' and $H(1-\chi)([\frac{1}{q_\varepsilon^+}]^\vee * \Psi_R) \to H(1-\chi)([\frac{1}{q^+}]^\vee * \Psi_R)$ in S'. □

Finally we have

PROPOSITION 4.5.23. *Let* f^+ *be given by equation* (4.5.19). *Then* f^+ *is a solution for equation* (4.5.18).

PROOF. Lemma 4.5.21 shows that $f^+ \in M_\alpha \cap S'$ for $\alpha < 1$. So we must verify that $Ff^+ = \Psi_R$ on Γ_+.

1.
$$\begin{aligned}
\left[c_3\hat{f}^+\right]^\vee &\overset{(1)}{=} \check{c}_3 * \left(\left[\frac{1}{q^-}\right]^\vee * H(x)\left(\left[\frac{1}{q^+}\right]^\vee * \Psi_R\right)\right) \\
&\overset{(2)}{=} \check{c}_3 * \left(\left[\frac{1}{q^-}\right]^\vee * H(x)\left(\lim_{\varepsilon\downarrow 0}\left[\frac{1}{q_\varepsilon^+}\right]^\vee * \Psi_R\right)\right) \\
&\overset{(3)}{=} \lim_{\varepsilon\downarrow 0}\left(\check{c}_3 * \left(\left[\frac{1}{q^-}\right]^\vee * H(x)\left(\left[\frac{1}{q_\varepsilon^+}\right]^\vee * \Psi_R\right)\right)\right) \\
&\overset{(4)}{=} \lim_{\varepsilon\downarrow 0}\left(\left(\check{c}_3 * \left[\frac{1}{q^-}\right]^\vee\right) * H(x)\left(\left[\frac{1}{q_\varepsilon^+}\right]^\vee * \Psi_R\right)\right).
\end{aligned}$$

(1): $\check{c}_3 \in \mathcal{O}_C'$.

(2): Proposition 4.5.15.

(3): Lemma 4.5.22, continuity of convolution between two distributions with support in $\overline{\mathbb{R}^+}$ and separate continuity of convolution between a distribution in $\mathcal{O}_C'$ and a distribution in S'.

(4): Lemma 4.5.22, associativity of convolution of distributions if all but one are elements of $\mathcal{O}'_C$.

2.
$$\left[\frac{Ai'(0)}{Ai(0)} e^{-\pi i/3} (-\xi+i0)^{\frac{2}{3}}\right]^\vee * f^+ =$$
$$= \lim_{\varepsilon\downarrow 0}\left(\left[\frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}(-\xi+i\varepsilon)^{\frac{2}{3}}\right]^\vee * \left(\left[\frac{1}{q^-}\right]^\vee * H(x)\left(\left[\frac{1}{q_\varepsilon^+}\right]^\vee * \Psi_R\right)\right)\right)$$
$$= \lim_{\varepsilon\downarrow 0}\left(\left[\frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}(-\xi+i\varepsilon)^{\frac{2}{3}} \frac{1}{q^-}\right]^\vee * H(x)\left(\left[\frac{1}{q_\varepsilon^+}\right]^\vee * \Psi_R\right)\right).$$

3.
$$H(x)\left(\left[\frac{1}{q_\varepsilon^+}\right]^\vee * \Psi_R\right) = \left[\frac{1}{q_\varepsilon^+}\right]^\vee * \Psi_R - H(-x)\left(\left[\frac{1}{q_\varepsilon^+}\right]^\vee * \Psi_R\right),$$

and both terms on the right are elements of $\mathcal{O}'_C$.

$$\check{c}_3 * \left[\frac{1}{q^-}\right]^\vee + \left[\frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}(-\xi+i\varepsilon)^{\frac{2}{3}} \frac{1}{q^-}\right]^\vee =$$
$$= \left[q^+\right]^\vee + \frac{Ai'(0)}{Ai(0)} e^{-\pi i/3}\left[\left((-\xi+i\varepsilon)^{\frac{2}{3}} - (-\xi+i0)^{\frac{2}{3}}\right)\frac{1}{q^-}\right]^\vee,$$

because we are dealing with elements of $\mathcal{S}'$ which have Fourier transforms that are continuous functions.

The second term on the right converges to 0 in $\mathcal{S}'$ and has support in $\overline{\mathbb{R}^+}$.

$$\left[q^+\right]^\vee * H(-x)\left(\left[\frac{1}{q_\varepsilon^+}\right]^\vee * \Psi_R\right)$$

is zero for $x > 0$ because both distributions have support in $\overline{\mathbb{R}^-}$.

Finally:
$$\left[q^+\right]^\vee * \left(\left[\frac{1}{q_\varepsilon^+}\right]^\vee * \Psi_R\right) = \left(\left[q^+\right]^\vee * \left[\frac{1}{q_\varepsilon^+}\right]^\vee\right) * \Psi_R$$
and
$$\left[q^+\right]^\vee * \left[\frac{1}{q_\varepsilon^+}\right]^\vee = \left[q^+ \frac{1}{q_\varepsilon^+}\right]^\vee, \quad q^+ \frac{1}{q_\varepsilon^+} \to 1 \quad \text{in } \mathcal{S}'$$
so
$$\left(\left[q^+\right]^\vee * \left[\frac{1}{q_\varepsilon^+}\right]^\vee\right) * \Psi_R \to \Psi_R \quad \text{in } \mathcal{S}'.$$

So indeed $Ff^+ = \Psi_R$ on Γ_+. □

<u>THEOREM 4.5.24</u>. *Let* f^+ *be given by equation* (4.5.19),

$$f := f^+ + f^-,$$
$$g := [c_3\hat{f} + c_4\hat{k}]^\vee.$$

Then $E^1(f,g)$ *is a solution for problem* (4.5.1) *except for the fact that* $E^1(f,g)|_\Gamma = h$ *only for* $x < \frac{R}{2}$. *This solution depends continuously on* f^-, h *and* k *in the sense that if* $k_j \to 0$ *in* E', $f_j^- \to 0$ *in* $E'(\Gamma_-)$, $h_j \to 0$ *in* $E'(\Gamma)$ *and for some* $\varepsilon > 0$ $\mathrm{supp}(f_j^-) \subset (-\infty,-\varepsilon]$ *and* $\mathrm{supp}(h_j) \subset [\varepsilon,\infty)$, *then* $E^1(f_j,g_j) \to 0$ *in* $\mathcal{D}'(\Omega)$.

PROOF. That $E^1(f,g)$ is such a solution follows from Proposition 4.5.5, Proposition 4.5.23 and

$$E^1(f,g)|_\Gamma = E(f^+,g)|_\Gamma = S(\psi_R Qh) = SQh + S((\psi_R - 1)Qh)$$
$$= h + S((\psi_R - 1)Qh).$$

See section 4.3.
Now $S((\psi_R - 1)Qh) = 0$ for $x < \frac{R}{2}$.
Under the conditions of the Theorem, $\Psi_{R,j} := \psi_R Qh_j - [c_3\hat{f}_j^- + c_4\hat{k}_j]^\vee \to 0$ in $\mathcal{O}_C'$ and the $\Psi_{R,j}$ are smooth for $|x| < \varepsilon$. But then for f_j^+ defined by formula (4.5.19), $f_j^+ \to 0$ in S'. Further:

$$\hat{g}_j = c_3(\hat{f}_j^- + \hat{f}_j^+) + c_4\hat{k}_j,$$

so for χ as in section 4.2, formula (4.2.11):

$$\chi\hat{g}_j = \frac{Ai'(0)}{Ai(0)}|\xi|^{\frac{2}{3}}\chi(\xi)(\hat{f}_j^- + \hat{f}_j^+) + \chi(\xi)\left[e^{-\frac{4}{3}|\xi|}\sigma_1(\xi)\widehat{f_j^- + f_j^+} + e^{-\frac{2}{3}|\xi|}\sigma_2(\xi)\hat{k}_j\right].$$

See Lemma 4.5.4.
Now

$$\chi(\xi)\frac{\pi Ai(0)}{|\xi|^{\frac{2}{3}}}\left[e^{-\frac{2}{3}|\xi|}\sigma_1(\xi)\widehat{f_j^- + f_j^+} + \sigma_2(\xi)\hat{k}_j\right] \to 0 \quad \text{in } S'.$$

Therefore $f_j^+ + f_j^-$ and g_j satisfy the condition in Proposition 4.2.13, part 2. Then Proposition 4.2.13 part 5 shows the continuity. □

Choose R_0 so large that $WF(\psi_{R_0}Qh) = WF(Qh)$. This is possible as can be derived from Remark 4.3.12. Then the singularities of f^+ for $x > 0$ are those of Qh. See Lemma 4.5.21. Since Theorem 4.5.24 gives a solution in Ω this implies that: h is singular in $(x,\xi) \Leftrightarrow E^1(f,g)$ is singular along the strip through $(2x,0,\xi,0)$. See again Remark 4.3.12.

Moreover, if we choose $R_1 > R_0$ then $f_{R_1}^+ - f_{R_0}^+$ is smooth for $x > 0$ so the solutions obtained differ by a smooth function in Ω.

Finally note that $(1 - \psi_{R_0})Qh$ is smooth. So is $S((1 - \psi_{R_0})Qh)$. This is the error made on Γ. Let us show that this error can be made arbitrarily small in the supremum norm.

PROPOSITION 4.5.25. *Let* $h \in E'(\Gamma)$, $\mathrm{supp}(h) \subset [\varepsilon, M-1]$, $\varepsilon > 0$, $M < \infty$. *Choose* R_0 *so that* $R_0 > 2M$, $\frac{1}{R_0} < \frac{\varepsilon}{2}$. *Then for* $R > R_0$:

1) $S((1-\psi_R)Qh)$ *is smooth on* Γ.

2) $\sup_{x\in\Gamma} |S((1-\psi_R)Qh)| \to 0$ *for* $R \to \infty$.

PROOF. The first part was made clear above. As to the second part, note that h can be written as

$$h = \sum_{k=1}^{m} \frac{\partial^{m_k}}{\partial x^{m_k}} f_k, \quad m < \infty,\ m_k < \infty,\ f_k \text{ continuous},$$
$$\mathrm{supp}\ f_k \subset [\tfrac{\varepsilon}{2}, M],\ k = 1,\dots,m.$$

Since $Qh = \mathrm{const}\cdot x_+^{\frac{1}{6}}(x_+^{-\frac{11}{6}} * h(\frac{x}{2}))$ it therefore suffices to show that

$$\sup_{x\in\Gamma} |S((1-\psi_R)x_+^{\frac{1}{6}}(x_+^{\alpha} * f))| \to 0 \quad \text{for } R \to \infty,\ \alpha \le -\tfrac{11}{6},\ f \text{ continuous},$$
$$\mathrm{supp}(f) \subset [\varepsilon, 2M].$$

The next inequalities can easily be derived. For $x \ge R$

$$|x_+^{\alpha} * f| \overset{!}{=} \left|\int_0^{2M} f(y)(x-y)^{\alpha}dy\right| \le C(x-2M)^{\alpha} \le C(R-2M)^{\alpha}(\tfrac{x}{R})^{\beta}$$

for all $\beta \ge \alpha$. Here C only depends on f, not on R. So formula (4.3.9) shows that for $x < \frac{R}{2}$:

$$S((1-\psi_R)x_+^{\frac{1}{6}}(x_+^{\alpha} * f)) = 0$$

and for $x \ge \frac{R}{2}$:

$$|S((1-\psi_R)x_+^{\frac{1}{6}}(x_+^{\alpha} * f))| = \mathrm{const}\left|\int_0^{2x} \frac{(1-\psi_R)(s)(x_+^{\alpha} * f)(s)}{(2x-s)^{\frac{1}{6}}}ds\right|$$
$$\le \mathrm{const}\cdot\frac{(R-2M)^{\alpha}}{R^{\beta}}\int_0^{2x}\frac{s^{\beta}}{(2x-s)^{\frac{1}{6}}}ds = C\frac{(R-2M)^{\alpha}}{R^{\beta}}(2x)^{\beta-\frac{1}{6}+1}$$

provided $\beta > -1$. Here we used $x_+^p * x_+^q = x_+^{p+q+1}$. Choose $-1 < \beta < -\frac{5}{6}$. Note that $\alpha \le -\frac{11}{6}$ so this is allowed. Then

$$\sup_{x\in\Gamma} |S((1-\psi_R)x_+^{\frac{1}{6}}(x_+^{\alpha} * f))| \le C\cdot\frac{(R-2M)^{\alpha}}{R^{\beta}}R^{\beta+\frac{5}{6}} = C(R-2M)^{\alpha}R^{\frac{5}{6}} \to 0$$

for $R \to \infty$ because $\alpha + \frac{5}{6} < 0$. □

CHAPTER 5

THE PSEUDO TRICOMI OPERATOR

5.1. Introduction.

In this chapter we discuss the PDO P_α given by

$$P_\alpha = t\frac{\partial^2}{\partial t^2} + \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + \alpha\frac{\partial}{\partial t}.$$

Here $\alpha \in \mathbb{C}$ is a constant.
We call this operator the Pseudo Tricomi operator for its resemblance to the Tricomi operator T (at least at first sight). P_α is elliptic for $t > 0$ and hyperbolic for $t < 0$ as T is. However the set $\{(x,t) \mid t = 0\}$ is characteristic. This has as consequence that P_α is not of real principal type in $\mathbb{R}^{n+1}$. See section 5.2. Therefore existence of solutions of $P_\alpha u = f$ and smoothness properties near $t = 0$ are open questions.

The operator P_α for $n = 1$ appeared in a paper by I.L. Karol ([17]). In this paper boundary value problems for $P_\alpha u = 0$ were investigated in a bounded region with boundary consisting for $t < 0$ of parts of bicharacteristic curves (cf. section 5.8, fig. 21). The number of boundary values which can be prescribed on these curves proves to be dependent on α. We will hardly discuss boundary value problems for P_α but we come back to this dependence on α in section 5.8.
We will construct a fundamental solution for P_α, which has smoothness properties that can easily be seen to be dependent on α.

For every x the set $\{(x,0,0,\tau) \mid \tau \gtrless 0\}$ describes a bicharacteristic strip of P_α. See section 5.2. Such a strip has the direction of the cone axis through $(x,0,0,\pm 1)$ (see section 2.12). These strips are responsible for the difficulties that appear because P_α is not of real principal type.

V. Guillemin and D. Schaëffer ([13]) discussed a class of PDOs $P_{(\alpha,\beta)}$ having strips which are equal to a cone axis. In this class P_α appears for $\beta = 0$. They discussed the qualitative properties of u such that $P_\alpha u \in C^\infty$ in the neighbourhood of such a strip. However they only considered the values of (α,β) so that $P_{(\alpha,\beta)}$ has only one strip equal to a cone axis. So P_α was not included in their discussion.

It turns out that the fact that P_α is independent of x causes serious problems in describing the singularities of solutions of $Pu = f$ for $t = 0$. On the other hand, this fact enables us to give solutions for arbitrary $f \in \mathcal{E}'$.

5.2. The bicharacteristic relation.

Let us first determine the bicharacteristic strips of P_α. For all α the principal symbol of P_α is given by

$$-t\tau^2 - |\xi|^2,$$

so the Hamilton-Jacobi equations for these strips become

$$(5.2.1)\qquad \frac{dx_j}{ds} = -2\xi_j,\quad \frac{d\xi_j}{ds} = 0,\quad j = 1,\dots,n,\quad \frac{dt}{ds} = -2t\tau,\quad \frac{d\tau}{ds} = \tau^2$$

under the condition $(-t\tau^2 - |\xi|^2)(s) = 0$.

Note that the system (5.2.1) is degenerate for $\xi = 0$, $t = 0$, $\tau \neq 0$. This already indicates that we can expect trouble at $t = 0$. The strip that starts for $s = 0$ in (x_0,t_0,ξ_0,τ_0), $t_0\tau_0^2 + |\xi_0|^2 = 0$, can be obtained as follows.

Because $\tau = 0$, $t\tau^2 + |\xi|^2 = 0$ implies $\xi = 0$ we have $\tau(s) \neq 0$ along each strip. In particular $\tau_0 \neq 0$.
Then $d\tau/ds = \tau^2$ gives $\tau(s) = \tau_0/(1 - s\tau_0)$, $s\tau_0 < 1$.
Because $\xi(s) \equiv \xi_0$ one easily gets that the strip is given by

$$(5.2.2)\qquad \left\{\left(x_0 - 2\xi_0 s,\ -(1 - s\tau_0)^2\frac{|\xi_0|^2}{\tau_0^2},\ \xi_0,\ \frac{\tau_0}{1-s\tau_0}\right) \mid s\tau_0 < 1\right\}.$$

For $\xi_0 = 0$ this gives the halfray with direction $(0,\tau_0)$ above $(x_0,0)$. So we have

PROPOSITION 5.2.3. *Let Ω be an open set in $\mathbb{R}^{n+1}$. Then P_α is of real principal type in Ω if and only if Ω doesnot contain points (x,t) with $t = 0$.* □

By writing $\tau = \tau_0/(1-s\tau_0)$ we can also describe the strip through (x_0,t_0,ξ_0,τ_0) as

$$(5.2.4)\qquad \{(x_0 - \frac{2\xi_0}{\tau_0} + \frac{2\xi_0}{\tau},\ -\frac{|\xi_0|^2}{\tau^2},\ \xi_0,\ \tau)\,|\,\tau\tau_0 > 0\}.$$

From this we can derive that the bicharacteristic relation C of P_α, P_α considered on the whole $\mathbb{R}^{n+1}$, is given by

$$\{((x_0 - \frac{2\xi_0}{\tau_0} + \frac{2\xi_0}{\tau}, -\frac{|\xi_0|^2}{\tau^2}, \xi_0, \tau),(x_0, t_0, \xi_0, \tau_0))\,|\, t_0\tau_0^2 + |\xi_0|^2 = 0,\ \tau\tau_0 > 0\}$$

$$= \{((y - \frac{2\xi}{\sigma} + \frac{2\xi}{\tau}, -\frac{|\xi|^2}{\tau^2}, \xi, \tau),(y, -\frac{|\xi|^2}{\sigma^2}, \xi, \sigma))\,|\,\sigma\tau > 0\}.$$

C is a C^∞-submanifold of $T^*(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})$.
However, C is not closed in $T^*(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})\setminus 0$. This is a consequence of the fact that a strip given by expression (5.2.4) converges to the strip $\{(x_0 - 2\xi_0/\tau_0, 0, 0, \tau)\,|\,\tau\tau_0 > 0\}$ in the sense that for $\tau\tau_0 \to \infty$, τ_0 fixed,

$$\left(x_0 - \frac{2\xi_0}{\tau_0} + \frac{2\xi_0}{\tau},\ -\frac{|\xi_0|^2}{\tau^2}\right) \to (x_0 - \frac{2\xi_0}{\tau_0}, 0)$$

$$(|\xi_0|^2 + \tau^2)^{-\frac{1}{2}}(\xi_0,\tau) \to \left(0, \frac{\tau_0}{|\tau_0|}\right) \qquad \text{(i.e. normalized).}$$

<u>PROPOSITION 5.2.5</u>. *The closure of* C *in* $T^*(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})\setminus 0$ *is given by* $Cl(C) = C \cup C_1 \cup C_2$ *with*

$$C_1 = \{((x,0,0,\tau),(y, -\tfrac{1}{4}|x-y|^2,0,0))\,|\,\tau \neq 0\}$$

$$C_2 = \{((x,-\tfrac{1}{4}|x-y|^2,0,0),(y,0,0,\sigma))\,|\,\sigma \neq 0\}.$$

<u>PROOF</u>. Consider $(c_j)_{j\in\mathbb{N}} \subset C$, $c_0 \in T^*(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})\setminus 0$ with

$$c_j = ((y_j - \frac{2\xi_j}{\sigma_j} + \frac{2\xi_j}{\tau_j}, -\frac{|\xi_j|^2}{\tau_j^2}, \xi_j, \tau_j),(y_j, -\frac{|\xi_j|^2}{\sigma_j^2}, \xi_j, \sigma_j)),\ \sigma_j\tau_j > 0,$$

$$c_0 = ((x_0, t_0, \xi_0, \tau_0),(y_0, s_0, \eta_0, \sigma_0)),\ |\xi_0|^2 + \tau_0^2 + |\eta_0|^2 + \sigma_0^2 \neq 0,$$

$$c_j \to c_0.$$

If $\tau_0\sigma_0 > 0$ then evidently $c_0 \in C$. If $\tau_0 = 0$, $\sigma_0 = 0$ then necessarily $\xi_0 = 0$, $\eta_0 = 0$, but this is not allowed. If $\tau_0 = 0$, $\sigma_0 \neq 0$ then also necessarily $\xi_0 = \eta_0 = 0$. So $s_0 = 0$ and

$$t_0 = \lim -\frac{|\xi_j|^2}{\tau_j^2} = -\tfrac{1}{4}\lim\left|y_j - \frac{2\xi_j}{\sigma_j} + \frac{2\xi_j}{\tau_j} - y_j\right|^2 = -\tfrac{1}{4}|x_0-y_0|^2,$$

so $c_0 \in C_2$.
In the same way, if $\tau_0 \neq 0$ and $\sigma_0 = 0$ then $c_0 \in C_1$.

So $Cl(C) \subset C \cup C_1 \cup C_2$. The other inclusion is obvious. □

When dealing with an operator P of real principal type, on pseudoconvex sets Ω parametrices can be constructed with kernels having their wave front set in their bicharacteristic relation C (see section 2.12). Now wave front sets are closed while for P_α C is not. So we must expect that the sets C_1 and C_2 will play a significant role in the construction of parametrices for P_α. They represent the difficulties when P_α is not of real principal type.

(5.2.6) Taking account of formula (2.7.3) we might expect that C_1 will be met in the process of constructing parametrices E such that $EC_0^\infty \subset C^\infty$. Paragraph (2.7.5) and the set C_2 point to difficulties that arise, when Eu is to be defined for all $u \in \mathcal{E}'(\mathbb{R}^{n+1})$.

Phase functions defining C can easily be given. From formula (5.2.4) it is clear that

$$\varphi(x,t,y,s,\xi,\tau,\sigma) := \langle x-y,\xi\rangle + t\tau - s\sigma - \frac{|\xi|^2}{\tau} + \frac{|\xi|^2}{\sigma}$$

is such a function. But from Lemma A.7.2 it is also clear that the following functions define C for $t < 0$, $s < 0$, $\tau \gtrless 0$, $\sigma \gtrless 0$:

$$\varphi_\pm(x,y,t,s,\theta) := \langle x-y,\theta\rangle \mp 2((-t)^{\frac{1}{2}} - (-s)^{\frac{1}{2}})|\theta|.$$

Finally we look at the bicharacteristic curves of P_α. Because these curves are the projections of the bicharacteristic strips to the (x,t)-space, it follows from formula (5.2.2) that for $t_0 < 0$ a curve through (x_0,t_0) is given by

$$\{(x_0 - 2\xi(1-s),\ -s^2|\xi|^2) \mid s > 0\}$$

with ξ fixed such that $-t_0 = |\xi|^2$.
For a point (x,t) on such a curve we have

$$t = -s^2|\xi|^2 = -\tfrac{1}{4}|x - x_0 + 2\xi|^2,$$

so a curve lies on a (generalized) paraboloid with top $(x_0-2\xi,0)$. These paraboloids are degenerate characteristic conoids of P_α.

For $n = 1$ we have $\xi = \sqrt{-t_0}$ or $\xi = -\sqrt{-t_0}$.
The equations for the bicharacteristic curves then become

$$x - x_1 = 2\sqrt{-t} \quad \text{or} \quad x - x_1 = -2\sqrt{-t}, \quad t < 0.$$

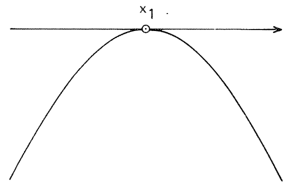

Fig. 19: bicharacteristic curves of P_α for n = 1.

Note that $(x_1,0)$ is the projection of a strip $\{(x_1,0,0,\tau) \mid \tau > 0\}$. So each parabola is the union of three bicharacteristic curves (if we call $\{(x_1,0)\}$ a curve, too).

REMARK 5.2.7. The set $\{(x,-\frac{1}{4}|x-y|^2) \mid x \in \mathbb{R}^n\}$ appearing in Proposition 5.2.5 is a degenerate characteristic conoid with top in (y,0).

5.3. Special solutions for the homogeneous equation.

On $\mathbb{R}^{n+1}$ we examine the equation

$$P_\alpha u = \left(t\frac{\partial^2}{\partial t^2} + \Delta_x + \alpha\frac{\partial}{\partial t}\right)u = 0.$$

Formal Fourier transformation with respect to x transforms this equation into

(5.3.1) $\quad \tilde{P}_\alpha\tilde{u} = 0$ with $\tilde{u} = \tilde{u}(\xi,t)$, $\tilde{P}_\alpha := t\frac{\partial^2}{\partial t^2} + \alpha\frac{\partial}{\partial t} - |\xi|^2.$

Further Fourier transformation with respect to t gives the equation

(5.3.2) $\quad \hat{P}_\alpha\hat{u} = 0$ with $\hat{u} = \hat{u}(\xi,\tau)$, $\hat{P}_\alpha := -i\tau^2\frac{\partial}{\partial \tau} - (2-\alpha)i\tau - |\xi|^2.$

For each of these equations we will give solutions defined on (open subsets of) $\mathbb{R}^{n+1}$. These solutions will be used in the next sections.

I. $\hat{P}_\alpha\hat{u} = 0$.

For $\tau \neq 0$ a solution is given by

$$c(\xi)\tau^{\alpha-2}\exp\left(-i\frac{|\xi|^2}{\tau}\right), \quad c \text{ arbitrary.}$$

Applying formally Fourier's inversion formula, we get a phase function $\langle x,\xi\rangle + t\tau - |\xi|^2/\tau$, which is familiar to us. Note that if $c(\xi)$ is considered to be the Fourier transform of some $u \in \mathcal{E}'(\mathbb{R}^n_x)$ then we get the formal solutions

$$(5.3.3) \qquad \int e^{i[\langle x-y,\xi\rangle + t\tau - |\xi|^2/\tau]}\tau^{\alpha-2}u(y)dyd\xi d\tau.$$

Of course we should be careful about integration near $\tau = 0$ (as well). The function $\varphi = \langle x-y,\xi\rangle + t\tau - |\xi|^2/\tau$ is a welldefined phase function outside a conic neighbourhood of $\tau = 0$.
However

$$\Lambda_\varphi = \{(y + \frac{2\xi}{\tau}, -\frac{|\xi|^2}{\tau^2}, \xi, \tau,; y, -\xi) \mid \tau \neq 0\}.$$

So φ doesnot satisfy the conditions given in section 2.8, paragraph (2.8.6). Choose $\xi = 0$! So expression (5.3.3) doesnot define a FIO. Therefore even for smooth u this expression doesnot necessarily define a smooth function. It might be singular for $t = 0$. This fact is closely related to the appearance of the set C_1 in Proposition (5.2.5). It represents the fact that $t = 0$ itself is a characteristic surface.

II. $\tilde{P}_\alpha\tilde{u} = 0$.

Keep $\xi \neq 0$ fixed.
If we try $\tilde{u} = t^\beta v_\xi(t)$ we obtain for $\beta = \frac{1}{2}(1-\alpha)$ the following ordinary differential equation for $v := v_\xi$:

$$t^2v'' + tv' - [|\xi|^2t + \tfrac{1}{4}(1-\alpha)^2]v = 0.$$

This is an equation which can be reduced to the Bessel equation (see section A.1). For $t \neq 0$ solutions are given by:

$$v_\xi^{(1)}(t) = J_{1-\alpha}(2i|\xi|t^{\frac{1}{2}}) \text{ and } v_\xi^{(2)}(t) = J_{\alpha-1}(2i|\xi|t^{\frac{1}{2}}).$$

For $1-\alpha \in \mathbb{Z}$ $J_{1-\alpha}(z) = (-1)^{1-\alpha}J_{\alpha-1}(z)$, so $v^{(1)}$ and $v^{(2)}$ are not linear independent. Although for these values of α also two independent solutions can be given, we will assume $\alpha \in \mathbb{C}\setminus\mathbb{Z}$ in case we are in a situation in which we use the solutions of the differential equation given above. We make this restriction in order to avoid additional difficulties in computation and notation.
Then solutions of $\tilde{P}_\alpha w = 0$ for $t \neq 0$ are

$$\tilde{u}_\alpha^{(1)}(\xi,t) = t^{\frac{1}{2}(1-\alpha)}J_{1-\alpha}(2i|\xi|t^{\frac{1}{2}}) \text{ and } \tilde{u}_\alpha^{(2)}(\xi,t) = t^{\frac{1}{2}(1-\alpha)}J_{\alpha-1}(2i|\xi|t^{\frac{1}{2}}).$$

Note that in the definition of $t^{\frac{1}{2}(1-\alpha)}$ and $t^{\frac{1}{2}}$ we do not have to use the same logarithm. Using in each case a branch of the logarithm defined for $-\frac{\pi}{2} < \arg t < \frac{3\pi}{2}$, we obtain a solution $k_{\alpha-1}$ which is smooth in t (and ξ) from $\tilde{u}_\alpha^{(2)}$, given by:

$$k_{\alpha-1}(\xi,t) := (2i|\xi|t^{\frac{1}{2}})^{1-\alpha}J_{\alpha-1}(2i|\xi|t^{\frac{1}{2}}).$$

PROPOSITION 5.3.4. *For each* $\alpha \in \mathbb{C}$, $k_{\alpha-1}$ *can be continued analytically to* $\mathbb{C}^{n+1}$ *and then* $k_{\alpha-1}$ *is a solution of* $\tilde{P}_\alpha w = 0$ *on the entire* $\mathbb{R}^{n+1}$.

PROOF. The analyticity of $k_{\alpha-1}$ follows from the fact that $k_{\alpha-1}(\xi,t) = j_{\alpha-1}(2i|\xi|t^{\frac{1}{2}})$, because $j_{\alpha-1}(z)$ is analytic and even in z (see section A.1). With the branch of the logarithm as above, we have for real (ξ,t), $|t\xi| \neq 0$,

$$(5.3.5)\qquad \begin{aligned} &(2i|\xi|t^{\frac{1}{2}})^{1-\alpha} = (2i|\xi|)^{1-\alpha}t^{\frac{1}{2}(1-\alpha)}, \text{ so} \\ &k_{\alpha-1}(\xi,t) = (2i|\xi|)^{1-\alpha}t^{\frac{1}{2}(1-\alpha)}J_{\alpha-1}(2i|\xi|t^{\frac{1}{2}}) = (2i|\xi|)^{1-\alpha}\tilde{u}_\alpha^{(2)}(\xi,t). \end{aligned}$$

So $\tilde{P}_\alpha k_{\alpha-1} = 0$ for $|t\xi| \neq 0$, but then $\tilde{P}_\alpha k_{\alpha-1} = 0$ on the entire $\mathbb{R}^{n+1}$. □

Using $\tilde{u}_\alpha^{(1)}$ we can define several solutions of $\tilde{P}_\alpha w = 0$ on the entire $\mathbb{R}^{n+1}$. For $\beta \in \mathbb{C}\setminus\mathbb{Z}$, we define S_β to be the subset of $\mathcal{D}'(\mathbb{R})$ consisting of the four distributions $t_+^\beta, t_-^\beta, (t+i0)^\beta$ and $(t-i0)^\beta$.
For $a = a_\beta \in S_\beta$ let $a_{\beta+1}$ be the corresponding element in $S_{\beta+1}$.

REMARK 5.3.6. The elements of $S_{1-\alpha}$ are solutions of $\tilde{P}_\alpha w = 0$ for $\xi = 0$. So is $w = 1$. Tensoring these distributions with $\delta_{(\xi=0)} \in \mathcal{D}'(\mathbb{R}^n_\xi)$ or $(\partial/\partial\xi_j)\delta_{(\xi=0)}$ gives solutions of $\tilde{P}_\alpha w = 0$ on $\mathbb{R}^{n+1}$. We have $t\cdot a_\beta = a_{\beta+1}$, the corresponding element in $S_{\beta+1}$. This is obvious for $\beta > 0$ and follows for arbitrary $\beta \in \mathbb{C}\setminus\mathbb{Z}$ by analytic continuation.

PROPOSITION 5.3.7. *Let* u_0 *be a smooth solution of* $\tilde{P}_\alpha w = 0$ *on* $\mathbb{R}^{n+1}$. *Let* $a = a_{\alpha-1} \in S_{\alpha-1}$, *considered as an element of* $\mathcal{D}'(\mathbb{R}^{n+1})$ *by tensoring with* $\mathbb{1}_\xi$. *Then* au_0 *is a solution of* $\tilde{P}_{2-\alpha}w = 0$ *on* $\mathbb{R}^{n+1}$.

PROOF. Because u_0 is smooth, au_0 is welldefined.

$$\frac{\partial}{\partial t}au_0 = u_0\frac{\partial a}{\partial t} + a\frac{\partial u_0}{\partial t} = (\alpha-1)a_{\alpha-2}u_0 + a_{\alpha-1}\frac{\partial u_0}{\partial t}.$$

$$\frac{\partial^2}{\partial t^2}au_0 = (\alpha-1)(\alpha-2)a_{\alpha-3}u_0 + 2(\alpha-1)a_{\alpha-2}\frac{\partial u_0}{\partial t} + a_{\alpha-1}\frac{\partial^2 u_0}{\partial t^2}.$$

Because $ta_{\alpha-3} = a_{\alpha-2}$ and $ta_{\alpha-2} = a_{\alpha-1}$ (Remark 5.3.6) it easily follows that $\tilde{P}_{2-\alpha}(au_0) = 0$. □

COROLLARY 5.3.8. *For every* $a \in S_{1-\alpha}$: $a \cdot k_{1-\alpha}(\xi,t)$ *is a solution of* $\tilde{P}_\alpha w = 0$ *on* $\mathbb{R}^{n+1}$.

PROOF. $k_{1-\alpha}$ is smooth and satisfies $\tilde{P}_{2-\alpha}k_{1-\alpha} = 0$. Note that $1-\alpha = (2-\alpha)-1$. Because $(2-\alpha)-1 = 1-\alpha$ and $2-(2-\alpha) = \alpha$, Proposition 5.3.7 gives the desired result. □

REMARK 5.3.9.
1. $\tilde{P}_{2-\alpha} = {}^t\tilde{P}_\alpha$.
2. The essence of Proposition 5.3.7 is that we obtain solutions defined on the entire $\mathbb{R}^{n+1}$. Similar statements for $t \neq 0$ are known (see Karol [17]).

Still using the same branch of the logarithm we can write

$$\begin{aligned}\tilde{u}_\alpha^{(1)}(\xi,t) &= t^{\frac{1}{2}(1-\alpha)}(2i|\xi|t^{\frac{1}{2}})^{1-\alpha}k_{1-\alpha}(\xi,t)\\ &= (2i|\xi|)^{1-\alpha}(t+i0)^{1-\alpha}k_{1-\alpha}(\xi,t).\end{aligned}$$

Corollary 5.3.8 now says that we can "cut off" $\tilde{u}_\alpha^{(1)}$ at $t = 0$. This is not possible for $\tilde{u}_\alpha^{(2)} = (2i|\xi|)^{\alpha-1}k_{\alpha-1}$ because $\tilde{P}_\alpha(H(t)k_{\alpha-1}) = (\alpha-1)k_{\alpha-1}(\xi,0)\delta_{t=0} = (\alpha-1)j_{\alpha-1}(0)\delta_{t=0}$ and $j_{\alpha-1}(0) \neq 0$, $\alpha \notin \mathbb{Z}$.

III. $P_\alpha u = 0$.

A consequence of Remark 5.3.6 is that 1, x_j, $a_{1-\alpha}$ and $x_j a_{1-\alpha}$ are solutions of $P_\alpha u = 0$ for all $a_{1-\alpha} \in S_{1-\alpha}$. We will now seek solutions defined as $u(|x|^2/4+t)$, u some element of $\mathcal{D}'(\mathbb{R})$. Note that $|x|^2/4+t = 0$ gives a characteristic conoid: see Remark 5.2.7.

Consider the function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $f(x,t) = |x|^2/4+t$. f is smooth and Df is surjective everywhere. Therefore the pullback f^*u is welldefined for all $u \in \mathcal{D}'(\mathbb{R})$, mapping $\mathcal{D}'(\mathbb{R})$ into $\mathcal{D}'(\mathbb{R}^{n+1})$ continuously (see section 2.6). For continuous u

$$\begin{aligned}\langle f^*u,\varphi\rangle &= \iint dxdt\, u(f(x,t))\varphi(x,t) = \iint dxdt\, u\left(\frac{|x|^2}{4}+t\right)\varphi(x,t)\\ &= \iint dxds\, u(s)\varphi\left(x, s-\frac{|x|^2}{4}\right).\end{aligned}$$

So for $u \in \mathcal{D}'(\mathbb{R})$ f^*u can be defined by

$$\langle f^*u,\varphi\rangle := \langle u, \int \varphi\left(x, s-\frac{|x|^2}{4}\right)dx\rangle .$$

Here

$$(5.3.10) \qquad \varphi \to \int \varphi\left(x,\, s - \frac{|x|^2}{4}\right) dx$$

defines a continuous map between $C_0^\infty(\mathbb{R}^{n+1})$ and $C_0^\infty(\mathbb{R})$.

For later convenience let us prove the next proposition.

PROPOSITION 5.3.11. *Formula* (5.3.10) *defines a continuous map between* $S(\mathbb{R}^{n+1})$ *and* $S(\mathbb{R})$.

PROOF. Choose $\varphi \in S(\mathbb{R}^{n+1})$. Then

$$\forall \gamma:\ \forall N \geq 0:\ \exists C_{N,\gamma} < \infty:\ \sup |(1 + |x|^2 + t^2)^{\frac{N}{2}} D^\gamma \varphi(x,t)| < C_{N,\gamma}.$$

In particular

$$\left|\varphi\left(x,\, s - \frac{|x|^2}{4}\right)\right| \leq \frac{C_{n+1,0}}{(1 + |x|^2)^{(n+1)/2}}$$

so the integral is convergent. It is a smooth function in s because φ is smooth and similar estimates hold for $D_t^m \varphi(x,t)$ for all m.
Let us now show that the integral is rapidly decreasing in s. That its derivatives also decrease rapidly then follows by similar arguments.
A simple calculation shows that

$$1 + |x|^2 + \left(s - \frac{|x|^2}{4}\right)^2 \geq \begin{matrix} s^2 + 1 \\ 4s - 3 \end{matrix} \quad \text{for } s \begin{matrix} \leq 2 \\ \geq 2 \end{matrix}.$$

Because

$$\left|\varphi\left(x,\, s - \frac{|x|^2}{4}\right)\right| \leq \frac{C_{2m+n+1,0}}{(1 + |x|^2 + (s - |x|^2/4)^2)^{m+(n+1)/2}} \leq$$

$$\leq \frac{C_{2m+n+1,0}}{(1 + |x|^2)^{(n+1)/2}} \cdot \frac{1}{(1 + |x|^2 + (s - |x|^2/4)^2)^m}$$

it follows that $s^m \int \varphi(x, s - |x|^2/4)dx$ is bounded for all $m \geq 0$.
Conclusion: the integral defines an element of $S(\mathbb{R})$.
If now $\varphi_j \to 0$ in $S(\mathbb{R}^{n+1})$ then $\forall (\gamma, N)$: $C^j_{N,\gamma} \to 0$ $(j \to \infty)$. From the estimates above it is then clear that $\int \varphi_j(x, s - |x|^2/4)dx \to 0$ in $S(\mathbb{R})$ as well. □

COROLLARY 5.3.12. *If* $u \in S'(\mathbb{R})$ *then* $f^*u \in S'(\mathbb{R}^{n+1})$. □

If we compute $P_\alpha(f^*u)$ we get

$$P_\alpha(f^*u) = t f^* \frac{d^2u}{ds^2} + \alpha f^* \frac{du}{ds} + \frac{n}{2} f^* \frac{du}{ds} + \frac{|x|^2}{4} f^* \frac{d^2u}{ds^2} \quad \text{(chainrule)}$$

$$= f^*\left(\left[s\frac{d^2}{ds^2} + \left(\alpha + \frac{n}{2}\right)\frac{d}{ds}\right]u\right).$$

So we obtain a solution of $P_\alpha v = 0$ if we choose $v = f^*u$ with u a solution of

$$\left[s\frac{d^2}{ds^2} + \left(\alpha + \frac{n}{2}\right)\frac{d}{ds}\right]u = 0.$$

(5.3.13)

For $\alpha + \frac{n}{2} \notin \mathbb{Z}$ we can take $u \in S_{1-\alpha-n/2}$ (or $u = 1$).

For $\alpha + \frac{n}{2} \in \mathbb{Z}$, $\alpha + \frac{n}{2} \le 0$, $u \in \{s_+^{1-m}, s_-^{1-m}, s^{1-m}, 1\}$, $m = \alpha + \frac{n}{2}$.

For $\alpha + \frac{n}{2} = 1$, $u \in \{H(s), H(-s), \ln(s+i0), \ln(s-i0), 1\}$.

For $\alpha + \frac{n}{2} \in \mathbb{Z}$, $\alpha + \frac{n}{2} \ge 2$, $u \in \{\delta_{s=0}^{(m-2)}, (s+i0)^{1-m}, (s-i0)^{1-m}, 1\}, m = \alpha + \frac{n}{2}$.

Note that for n even, automatically $\alpha + \frac{n}{2} \notin \mathbb{Z}$ if $\alpha \notin \mathbb{Z}$.

5.4. A non-smoothness result.

For ψDOs P of real principal type, $u \in E'$, $Pu \in C^\infty$ implies $u \in C^\infty$. In this section we show that P_α does not have this property when considering it on (open) subsets of $\mathbb{R}^{n+1}$ containing points (x,t) with $t = 0$ (see also Proposition 5.2.3).
Also we show a consequence this fact has for the qualitative properties of a parametrix for P_α.

PROPOSITION 5.4.1. *Let* $\varepsilon > 0$ *be given. There exists a distribution* $u \in E'(\mathbb{R}^{n+1})$ *so that*

1. $P_\alpha u \in C^\infty$.
2. $\mathrm{supp}(u) \in \{(x,t) \mid |x|^2 + t^2 \le \varepsilon^2\}$.
3. $u \notin C^\infty$.

PROOF. Choose $\varphi \in C_0^\infty(\mathbb{R}^n)$ so that $\varphi(x) = \begin{cases} 1 & \text{for } |x| < \varepsilon/4 \\ 0 & \text{for } |x| > \varepsilon/2 \end{cases}$.
Let $b_\alpha := s_-^{1-\alpha-n/2}$, $\alpha + \frac{n}{2} \ne 2,3,4,\ldots$, $\alpha \ne 2,3,4,\ldots$. We will give the proof only for these values of α. For $\alpha + \frac{n}{2} = 2,3,4,\ldots$, one should choose $b_\alpha = \delta_{s=0}^{\alpha+(n/2)-2}$, for $\alpha = 2,3,4,\ldots$, n odd, $b_\alpha = (s \pm i0)^{1-\alpha-n/2}$.
Let f be as in the previous section. We define $v := \varphi *_x f^*(b_\alpha)$ (see section 2.5 for $*_x$). This is welldefined.
The results given in section 2.6 show that

$$WF(f^*(b_\alpha)) \subset \{(x, -\frac{|x|^2}{4}, \frac{\tau}{2}x, \tau) \mid \tau \ne 0\}.$$

But then Lemma A.7.5 gives:

(5.4.2) $\quad WF(\varphi *_x (f^*(b_\alpha))) \subset \{(x,0,0,\tau) \mid x \in \operatorname{supp}(\varphi)\}.$

Note that $P_\alpha(f^*b_\alpha) = 0$. Corollary 5.3.12 shows that $f^*b_\alpha \in S'(\mathbb{R}^{n+1})$. So $\hat{P}_\alpha(\widehat{f^*b_\alpha}) = 0$, too. But then $\hat{P}_\alpha(\hat{\varphi}(\xi)\widehat{f^*b_\alpha}) = 0$ so v satisfies $P_\alpha v = 0$.
Let us show now that (0,0,0,1) and (0,0,0,-1) are in WF(v).
For $\alpha + \frac{n}{2} < 0$ it is easy to check that for $t < 0$

(5.4.3) $\quad \varphi *_x f^*(b_\alpha) = (-2i)^n t^{1-\alpha} \langle (1-|y|^2)_+^{1-\alpha-n/2}, \varphi(x - 2y\sqrt{-t})\rangle_y.$

Here $(1-\rho^2)_+^\beta$ is defined by analytic continuation for $\beta \neq -1,-2,\dots$. Then formula (5.4.3) holds for $\alpha + \frac{n}{2} \neq 2,3,4,\dots$ as well. From formula (5.4.2) it is clear that the restriction of v to $x = x_0$ is welldefined. For $t < 0$ it is given by formula (5.4.3) with $x = x_0$. Since $\varphi(-2y\sqrt{-t}) = 1$ for $|y| \le 1$ provided $2\sqrt{-t} < \frac{\varepsilon}{4}$ and

$$\int_0^1 \rho^{n-1}(1-\rho^2)_+^{1-\alpha-n/2} d\rho = \frac{\Gamma(n/2)\Gamma(2-\alpha-n/2)}{2\Gamma(2-\alpha)} \neq 0$$

for $\alpha \neq 2,3,4,\dots$, we see that v restricted to $x = 0$ behaves like $t^{1-\alpha}$ for $t \uparrow 0$. Now v is zero for $t > 0$. But then it follows straightforward that (0,1) and (0,-1) belong to the wave front set of the restriction of v. So $(0,0,0,\pm 1) \in WF(v)$. Choose $\psi \in C_0^\infty(\mathbb{R}^{n+1})$ so that

$$\psi(x,t) = \begin{cases} 1 & \text{for } \sqrt{|x|^2+t^2} < 3\varepsilon/4 \\ 0 & \phantom{\text{for } \sqrt{|x|^2+t^2}} > \varepsilon \end{cases}.$$

Then $u := \psi v$ satisfies conditions 1, 2 and 3. □

One might wonder if it would be possible to improve part 1 of Proposition 5.4.1 by constructing $u \in E'$ satisfying conditions 2 and 3 and $P_\alpha u = 0$. This is not the case, as follows from the next proposition.

PROPOSITION 5.4.4. *Let* $u \in E'$ *be such that* $P_\alpha u = 0$. *Then* $u = 0$.

PROOF. If $u \in E'$ and $P_\alpha u = 0$, then $\hat{u}$ is an analytic function such that $\hat{P}_\alpha \hat{u} = 0$. For ξ fixed, $\tau \neq 0$ we get $\hat{u}(\xi,\tau) = c(\xi)\tau^{\alpha-2}\exp(-i|\xi|^2/\tau)$. From this expression it is clear that $c(\xi) = 0$ for all ξ. So $\hat{u} = 0$ and therefore $u = 0$. □

An important consequence of Proposition 5.4.1 is

PROPOSITION 5.4.5. *Let* $\alpha \in \mathbb{C}$ *be given. There is no operator* E *such that*

1) $E : C_0^\infty(\mathbb{R}^{n+1}) \to C^\infty(\mathbb{R}^{n+1})$ *continuously*

2) ${}^tE : C_0^\infty(\mathbb{R}^{n+1}) \to C^\infty(\mathbb{R}^{n+1})$

3) $P_\alpha E = I + R$ *where* I *is the identity and* R *has a smooth kernel.*

PROOF. Suppose an operator satisfying these conditions does exist. Let $u \in \mathcal{E}'$ be such that $P_{2-\alpha}u \in C^\infty$. Then $P_{2-\alpha}u \in C_0^\infty$. So ${}^tE(P_{2-\alpha}u)$ is smooth. Now ${}^tE(P_{2-\alpha}u) = {}^tE({}^tP_\alpha u) = {}^t(P_\alpha E)u = u + {}^tRu$, so u is smooth. But this contradicts Proposition 5.4.1. □

REMARK. The fact that E maps C_0^∞ continuously to C^∞ implies that ${}^tE$ maps $\mathcal{E}'$ continuously to $\mathcal{D}'$.

5.5. The construction of a fundamental solution.

In this section we will construct for $\alpha \in \mathbb{C}\setminus\mathbb{Z}$ an operator E_α such that $E_\alpha : C_0^\infty(\mathbb{R}^{n+1}) \to C^\infty(\mathbb{R}^{n+1})$ continuously and $P_\alpha E_\alpha \varphi = \varphi = E_\alpha P_\alpha \varphi$ for all $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$. Then E_α is a fundamental solution for P_α. The procedure will be as follows.

First we will consider the equation $\tilde{P}_\alpha \tilde{u} = \tilde{f}$, $f \in C_0^\infty(\mathbb{R}^{n+1})$. By means of the method of variation of constants we obtain solutions, at least for $t \neq 0$, which behave "well" for $|\xi| \to \infty$. Next we define a solution which is valid in a neighbourhood of $t = 0$, too. In the next section we then show that this gives us an operator E_α with the desired properties.

So let $\alpha \in \mathbb{C}\setminus\mathbb{Z}$, $\xi \neq 0$, $f \in C_0^\infty(\mathbb{R}^{n+1})$ and consider the equation

$$(5.5.1) \qquad \tilde{P}_\alpha \tilde{u} = \tilde{f}, \quad \tilde{f} = \tilde{f}(\xi,t).$$

Let $\tilde{u}_\alpha^{(1)}$ and $\tilde{u}_\alpha^{(2)}$ be the solutions of the homogeneous equation obtained in section 5.3. For the reason explained in that section, we choose the argument of t for the definition of $\tilde{u}_\alpha^{(2)}$ such that $\tilde{u}_\alpha^{(2)}$ becomes smooth in t. Therefore we take the branch of the logarithm with argument arg(t) such that $-\frac{\pi}{2} < \arg t < \frac{3\pi}{2}$. (Other choices for the argument will give the same function multiplied by a constant.) If we apply the same branch for the definition of $\tilde{u}_\alpha^{(1)}$, any other choice for the argument gives a solution which can be written as

$$\lambda H(t)\tilde{u}_\alpha^{(1)} + \mu H(-t)\tilde{u}_\alpha^{(1)}, \quad \lambda \neq 0 \text{ and } \mu \neq 0.$$

With $\lambda = 1$ we define this to be U_1 and we define $U_2 := \tilde{u}_\alpha^{(2)}$.

With $\tilde{v} := \frac{\partial \tilde{u}}{\partial t}$, equation (5.5.1) can be written as

$$(5.5.2)\qquad \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -|\xi|^2 & \alpha \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{f} \end{pmatrix}.$$

Solutions of this system for $\tilde{f} = 0$ are $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = c_1 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + c_2 \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$ with $u_i = U_i$, $v_i = \frac{\partial}{\partial t} U_i$, $c_i = c_i(\xi)$, $i = 1,2$.
If we now write $c_i := c_i(t,\xi)$, $i = 1,2$, (variation of constants) and substitute $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$ in equation (5.5.2) we get the equation

$$\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} \partial c_1/\partial t \\ \partial c_2/\partial t \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{f} \end{pmatrix}.$$

For $t \neq 0$ therefore

$$\begin{pmatrix} \partial c_1/\partial t \\ \partial c_2/\partial t \end{pmatrix} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^{-1} \begin{pmatrix} v_2 & -u_2 \\ -v_1 & u_1 \end{pmatrix} \begin{pmatrix} 0 \\ (1/t)\tilde{f} \end{pmatrix}.$$

Now

$$(5.5.3)\qquad \begin{vmatrix} t^{\frac{1}{2}(1-\alpha)} J_{1-\alpha}(2i|\xi|t^{\frac{1}{2}}) & t^{\frac{1}{2}(1-\alpha)} J_{\alpha-1}(2i|\xi|t^{\frac{1}{2}}) \\ \frac{\partial}{\partial t}\left[\quad " \quad\right] & \frac{\partial}{\partial t}\left[\quad " \quad\right] \end{vmatrix}$$

$$= t^{1-\alpha}(i|\xi|t^{-\frac{1}{2}}) \left. \begin{vmatrix} J_{1-\alpha}(z) & J_{\alpha-1}(z) \\ J'_{1-\alpha}(z) & J'_{\alpha-1}(z) \end{vmatrix} \right|_{z=2i|\xi|t^{\frac{1}{2}}}$$

$$= t^{1-\alpha}(i|\xi|t^{-\frac{1}{2}}) \frac{2 \sin (1-\alpha)\pi}{-2\pi i|\xi|t^{\frac{1}{2}}} = t^{-\alpha} \frac{\sin (1-\alpha)\pi}{-\pi}.$$

So for $t \neq 0$

$$\frac{\partial c_1}{\partial t} = \frac{\pi}{\sin (1-\alpha)\pi} (H(t) + \mu H(-t))^{-1} t^{\alpha-1} U_2(\xi,t)\tilde{f}(\xi,t),$$

$$\frac{\partial c_2}{\partial t} = \frac{-\pi}{\sin (1-\alpha)\pi} (H(t) + \mu H(-t))^{-1} t^{\alpha-1} U_1(\xi,t)\tilde{f}(\xi,t).$$

<u>REMARK</u>. $\partial c_2/\partial t$ is independent of μ while $\partial c_1/\partial t$ is not.

For later convenience we remark that if we choose $u_1 = \lambda_1 U_1 + \lambda_2 U_2$ and $u_2 = \lambda_3 U_1 + \lambda_4 U_2$, $\lambda_i = \lambda_i(\xi)$, $i = 1,2,3,4$, then

$$\frac{\partial c_1}{\partial t} = \frac{\pi}{\sin (1-\alpha)\pi} (H(t) + \mu H(-t))^{-1} \begin{vmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{vmatrix}^{-1} t^{\alpha-1} (\lambda_3 U_1 + \lambda_4 U_2)\tilde{f},$$

$$\frac{\partial c_2}{\partial t} = \frac{-\pi}{\sin (1-\alpha)\pi} (H(t) + \mu H(-t))^{-1} \begin{vmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{vmatrix}^{-1} t^{\alpha-1} (\lambda_1 U_1 + \lambda_2 U_2)\tilde{f}.$$

Let $\int w(\xi,s)ds$ denote a primitive of $w(\xi,t)$ with respect to t. Then we obtain for equation (5.5.1) the formal solution(s):

$$(5.5.4)\qquad \tilde{u}(\xi,t) = \frac{-\pi}{\sin(1-\alpha)\pi}\begin{vmatrix}\lambda_1 & \lambda_2\\ \lambda_3 & \lambda_4\end{vmatrix}^{-1} \times$$

$$\Big\{-[\lambda_1 U_1(\xi,t)+\lambda_2 U_2(\xi,t)]\int s^{\alpha-1}(H(s)+\mu H(-s))^{-1}[\lambda_3 U_1(\xi,s)+\lambda_4 U_2(\xi,s)]\tilde{f}(\xi,s)ds$$

$$+[\lambda_3 U_1(\xi,t)+\lambda_4 U_2(\xi,t)]\int s^{\alpha-1}(H(s)+\mu H(-s))^{-1}[\lambda_1 U_1(\xi,s)+\lambda_2 U_2(\xi,s)]\tilde{f}(\xi,s)ds\Big\}.$$

The asymptotic expansion of the Bessel functions (see section A.4) shows that $J_{\pm(1-\alpha)}(2i|\xi|t^{\frac{1}{2}})$ is exponentially increasing for $t > 0$ fixed, $|\xi| \to \infty$. In order to be able to apply Fourier's inversion formula we therefore choose λ_3 and λ_4 so that $\lambda_3 U_1 + \lambda_4 U_2$ is rapidly decreasing for every $t > 0$ fixed, $|\xi| \to \infty$. This can be done uniquely modulo a multiplicative constant with

$$\lambda_3 = \frac{-e^{(\alpha-1)\pi i}}{i\sin(1-\alpha)\pi} \quad\text{and}\quad \lambda_4 = \frac{1}{i\sin(1-\alpha)\pi}.$$

Then

$$\lambda_3 U_1 + \lambda_4 U_2 = t^{\frac{1}{2}(1-\alpha)}\left[\frac{-e^{(\alpha-1)\pi i}J_{1-\alpha}(2i|\xi|t^{\frac{1}{2}}) + J_{\alpha-1}(2i|\xi|t^{\frac{1}{2}})}{i\sin(1-\alpha)\pi}\right]$$

$$= t^{\frac{1}{2}(1-\alpha)}H^{(1)}_{1-\alpha}(2i|\xi|t^{\frac{1}{2}}), \qquad t > 0.$$

For the same reason as above we must choose the "intervals of integration" in expression (5.5.4) for $t > 0$ as follows: in the first integral we integrate over $t < s < \infty$, in the second over $(-\infty <)\ s < t$. If furthermore the Bessel functions are expressed in terms of the function $k_{\alpha-1}$ from section 5.3, with these adaptations expression (5.5.4) becomes for $t > 0$:

$$(5.5.5)\qquad \tilde{u}(\xi,t) = \left(\frac{-\pi}{\sin(1-\alpha)\pi}\cdot\frac{1}{\lambda_1+\lambda_2 e^{(\alpha-1)\pi i}}\right) \times$$

$$\Big\{[\lambda_1(2i|\xi|)^{1-\alpha}t^{1-\alpha}k_{1-\alpha}(\xi,t)+\lambda_2(2i|\xi|)^{\alpha-1}k_{\alpha-1}(\xi,t)] \times$$

$$\int_t^{\infty}[-e^{(\alpha-1)\pi i}(2i|\xi|)^{1-\alpha}k_{1-\alpha}(\xi,s)+s^{\alpha-1}(2i|\xi|)^{\alpha-1}k_{\alpha-1}(\xi,s)]\tilde{f}(\xi,s)ds$$

$$+[-e^{(\alpha-1)\pi i}t^{1-\alpha}(2i|\xi|)^{1-\alpha}k_{1-\alpha}(\xi,t)+(2i|\xi|)^{\alpha-1}k_{\alpha-1}(\xi,t)] \times$$

$$\int_0^{t}[\lambda_1(2i|\xi|)^{1-\alpha}k_{1-\alpha}(\xi,s)+\lambda_2 s^{\alpha-1}(2i|\xi|)^{\alpha-1}k_{\alpha-1}(\xi,s)]\tilde{f}(\xi,s)ds$$

$$+[-e^{(\alpha-1)\pi i}t^{1-\alpha}(2i|\xi|)^{1-\alpha}k_{1-\alpha}(\xi,t)+(2i|\xi|)^{\alpha-1}k_{\alpha-1}(\xi,t)] \times$$

$$\int_{(-\infty)}^{0}[\lambda_1(2i|\xi|)^{1-\alpha}k_{1-\alpha}(\xi,s)+\frac{\lambda_2}{\mu}s^{\alpha-1}(2i|\xi|)^{\alpha-1}k_{\alpha-1}(\xi,s)]\tilde{f}(\xi,s)ds\Big\}.$$

Omitting the first factor, this is linear in (λ_1,λ_2). For $\lambda_1 = 1$ and $\lambda_2 = 0$ we get:

$$(5.5.6)\qquad -e^{(\alpha-1)\pi i}(2i|\xi|)^{2(1-\alpha)}t^{1-\alpha}k_{1-\alpha}(\xi,t)\int_0^\infty k_{1-\alpha}(\xi,s)\tilde{f}(\xi,s)ds$$
$$+\ t^{1-\alpha}k_{1-\alpha}(\xi,t)\int_t^\infty s^{\alpha-1}k_{\alpha-1}(\xi,s)\tilde{f}(\xi,s)ds$$
$$+\ k_{\alpha-1}(\xi,t)\int_0^t k_{1-\alpha}(\xi,s)\tilde{f}(\xi,s)ds$$
$$+\left[-e^{(\alpha-1)\pi i}t^{1-\alpha}(2i|\xi|)^{1-\alpha}k_{1-\alpha}(\xi,t)+(2i|\xi|)^{\alpha-1}k_{\alpha-1}(\xi,t)\right]\ \times$$
$$\int_{(-\infty)}^0 (2i|\xi|)^{1-\alpha}k_{1-\alpha}(\xi,s)\tilde{f}(\xi,s)ds.$$

For $\lambda_1 = 0$ and $\lambda_2 = 1$ we get:

$$(5.5.7)\qquad (2i|\xi|)^{2(\alpha-1)}k_{\alpha-1}(\xi,t)\int_0^\infty s^{\alpha-1}k_{\alpha-1}(\xi,s)\tilde{f}(\xi,s)ds$$
$$-\ k_{\alpha-1}(\xi,t)\int_t^\infty e^{(\alpha-1)\pi i}k_{1-\alpha}(\xi,s)\tilde{f}(\xi,s)ds$$
$$-\ e^{(\alpha-1)\pi i}t^{1-\alpha}k_{1-\alpha}(\xi,t)\int_0^t s^{\alpha-1}k_{\alpha-1}(\xi,s)\tilde{f}(\xi,s)ds$$
$$+\frac{1}{\mu}\left[-e^{(\alpha-1)\pi i}t^{1-\alpha}(2i|\xi|)^{1-\alpha}k_{1-\alpha}(\xi,t)+(2i|\xi|)^{\alpha-1}k_{\alpha-1}(\xi,t)\right]\ \times$$
$$\int_{(-\infty)}^0 s^{\alpha-1}(2i|\xi|)^{\alpha-1}k_{\alpha-1}(\xi,s)\tilde{f}(\xi,s)ds.$$

Looking for an expression which is smooth in t for $t \downarrow 0$, we notice that expression (5.5.6) contains at least two singular terms, which donot cancel. In expression (5.5.7) only the last term is singular, but this term can be omitted since it is a solution of the homogeneous equation.

So finally we arrive at the following expression for $\tilde{u}$ for $t > 0$:

$$(5.5.8)\qquad \tilde{u}(\xi,t) = \frac{-\pi}{\sin(1-\alpha)\pi}\ \times$$
$$\Big\{e^{(1-\alpha)\pi i}(2i|\xi|)^{2(\alpha-1)}k_{\alpha-1}(\xi,t)\int_0^\infty s^{\alpha-1}k_{\alpha-1}(\xi,s)\tilde{f}(\xi,s)ds$$
$$-\ k_{\alpha-1}(\xi,t)\int_t^\infty k_{1-\alpha}(\xi,s)\tilde{f}(\xi,s)ds$$
$$-\ t^{1-\alpha}k_{1-\alpha}(\xi,t)\int_0^t s^{\alpha-1}k_{\alpha-1}(\xi,s)\tilde{f}(\xi,s)ds\Big\}.$$

For $t < 0$ we remark that the bicharacteristic structure of P_α more or less forces us to integrate over $s > t$ in expression (5.5.4). In order to get a smooth connection at $t = 0$ we therefore define $\tilde{u}(\xi,t)$ for $t < 0$ by expression (5.5.8), too.

We are now at the point of defining a fundamental solution for P_α. The interpretation of expression (5.5.8) is still not clear for two reasons. In the first place, the factor $|\xi|^{2(\alpha-1)}$ doesnot behave well at $\xi = 0$, although it might be interpreted as a distribution. However, the term containing it is a solution of the homogeneous equation, so we can multiply it by an arbitrary smooth function of ξ. Second we might have problems with the factors $s^{\alpha-1}$ and $t^{1-\alpha}$. Here we shall use the distributions $s_+^{\alpha-1}$ and $s_{[0,1]}^{\alpha-1}$ (see section 2.13) to overcome these difficulties.

Let $\psi(\xi)$ be a C^∞ function, $0 \le \psi \le 1$ and

$$\psi(\xi) = \begin{cases} 1 & \text{for } |\xi| > 1 \\ 0 & \text{for } |\xi| < \frac{1}{2} \end{cases}.$$

For $\alpha \in ¢ \setminus \mathbb{Z}$, $f \in C_0^\infty(\mathbb{R}^{n+1})$ define

$$(5.5.9) \qquad (F_\alpha f)(\xi,t) := \frac{-\pi}{\sin(1-\alpha)\pi} \times$$
$$\Big\{ e^{(1-\alpha)\pi i}\psi(\xi)(2i|\xi|)^{2(\alpha-1)}k_{\alpha-1}(\xi,t) <s_+^{\alpha-1}, k_{\alpha-1}(\xi,s)\tilde{f}(\xi,s)>$$
$$-k_{\alpha-1}(\xi,t)\int_t^\infty k_{1-\alpha}(\xi,s)\tilde{f}(\xi,s)ds$$
$$-tk_{1-\alpha}(\xi,t) <s_{[0,1]}^{\alpha-1}, k_{\alpha-1}(\xi,st)\tilde{f}(\xi,st)> \Big\}.$$

Here $< \, , \, >$ means (distributional) integration with respect to s.

$$(5.5.10) \qquad (E_\alpha f)(x,t) := \frac{1}{(2\pi)^n}\int e^{i<x,\xi>}(F_\alpha f)(\xi,t)d\xi.$$

<u>THEOREM 5.5.11</u>. *For* $\alpha \in ¢ \setminus \mathbb{Z}$, $f \in C_0^\infty(\mathbb{R}^{n+1})$ $E_\alpha f$ *is welldefined and smooth,* $E_\alpha : C_0^\infty(\mathbb{R}^{n+1}) \to C^\infty(\mathbb{R}^{n+1})$ *continuously and* $P_\alpha E_\alpha f = f = E_\alpha P_\alpha f$.

The (very technical) proof of this theorem will be given in the next section. From this theorem we can derive:

<u>COROLLARY 5.5.12</u>. *Let* E_α *be the operator of* Theorem 5.5.11. *Then* ${}^tE_\alpha$ *defines a continuous map between* $\mathcal{E}'(\mathbb{R}^{n+1})$ *and* $\mathcal{D}'(\mathbb{R}^{n+1})$ *by*

$$<{}^tE_\alpha u,\varphi> := <u,E_\alpha\varphi>.$$

It satisfies ${}^tE_\alpha P_{2-\alpha} = I = P_{2-\alpha}{}^tE_\alpha$ *on* $\mathcal{E}'(\mathbb{R}^{n+1})$.

PROOF. The proof follows easily from Theorem 5.5.11 and the fact that $P_{2-\alpha} = {}^tP_\alpha$. □

5.6. The proof of Theorem 5.5.11.

The analysis of E_α is far more difficult than the corresponding analysis of $A^{\pm}$ in Chapter 3 in the case of the Tricomi operator. For not only we have to deal with the exponential growth of some factors for $|\xi| \to \infty$, but also we have to be very careful with the distributions $s_+^{\alpha-1}$ and $s_{[0,1]}^{\alpha-1}$.

We will now analyse first the operator F_α. The results will be formulated in two propositions.

PROPOSITION 5.6.1. *For* $\alpha \in \mathbb{C}\setminus\mathbb{Z}$, $f \in C_0^\infty(\mathbb{R}^{n+1})$, $F_\alpha f$ *is welldefined and smooth in* (ξ,t) *and* $\tilde{P}_\alpha F_\alpha f = \tilde{f} = F_\alpha P_\alpha f$.

PROOF. $\psi(\xi)(2i|\xi|)^{2(\alpha-1)}$ is smooth in ξ. $k_\beta(\xi,t)$ is smooth in (ξ,t) for every $\beta \in \mathbb{C}$. $\tilde{f}(\xi,t)$ is smooth in (ξ,t) and if supp $f \subset \{(x,t) \mid |x|^2+t^2 \le R^2\}$, then $(D^\gamma_{(\xi,t)}\tilde{f})(\xi,t) = 0$ for all (ξ,t) with $|t| \ge R$, γ arbitrary. Now for $\alpha \in \mathbb{C}\setminus\mathbb{Z}$, $s_+^{\alpha-1} \in \mathcal{D}'(\mathbb{R})$ and $s_{[0,1]}^{\alpha-1} \in \mathcal{E}'(\mathbb{R})$, so it is clear that the integrals in expression (5.5.9) are welldefined for ξ fixed and smooth in (ξ,t). But then $F_\alpha f$ is also welldefined and smooth in (ξ,t).

We now show that $\tilde{P}_\alpha F_\alpha f = \tilde{f}$. This may seem obvious for $\alpha > 1$ and consequently for $\alpha \in \mathbb{C}\setminus\mathbb{Z}$, but is should be remarked, that the application of the method of variation of constants near $s = 0$ or $t = 0$ still needs some justification.
We choose the most simple way and compute $\tilde{P}_\alpha F_\alpha f$ directly. The first term of expression (5.5.9) is a solution of $\tilde{P}_\alpha u = 0$, because $k_{\alpha-1}$ is. Furthermore

$$\begin{aligned} t\frac{\partial}{\partial t}\langle s_{[0,1]}^{\alpha-1},\varphi(st)\rangle &= t\langle s_{[0,1]}^{\alpha-1}, s\varphi'(st)\rangle = \langle s_{[0,1]}^{\alpha}, t\varphi'(st)\rangle \\ &= \langle s_{[0,1]}^{\alpha}, \frac{\partial}{\partial s}\varphi(st)\rangle = -\langle \frac{d}{ds}s_{[0,1]}^{\alpha},\varphi(st)\rangle = \\ &= -\alpha\langle s_{[0,1]}^{\alpha-1},\varphi(st)\rangle + \varphi(t). \end{aligned}$$

Then it is easy to check that

$$\tilde{P}_\alpha F_\alpha f = \frac{-\pi}{\sin(1-\alpha)\pi}\left[-(1-\alpha)k_{1-\alpha}k_{\alpha-1} + tk_{1-\alpha}\frac{\partial}{\partial t}k_{\alpha-1} - tk_{\alpha-1}\frac{\partial}{\partial t}k_{1-\alpha}\right]\tilde{f} = \tilde{f},$$

as follows from expression (5.5.3).

Next: $F_\alpha P_\alpha f = \tilde{f}$.

$$\langle s_+^{\alpha-1}, k_{\alpha-1}(\xi,s)\widetilde{P_\alpha f}(\xi,s)\rangle = \langle {}^t\tilde{P}_\alpha(k_{\alpha-1}s_+^{\alpha-1}),\tilde{f}\rangle = 0$$

because of Proposition 5.3.7.

$$\int_t^\infty k_{1-\alpha}\widetilde{P_\alpha f}\,ds = \int_t^\infty \tilde{f}\,{}^t\tilde{P}_\alpha k_{1-\alpha}\,ds + (1-\alpha)k_{1-\alpha}\tilde{f} + t\tilde{f}\frac{\partial}{\partial t}k_{1-\alpha} - tk_{1-\alpha}\frac{\partial}{\partial t}\tilde{f}$$

and

$${}^t\tilde{P}_\alpha k_{1-\alpha} = \tilde{P}_{2-\alpha}k_{1-\alpha} = 0.$$

$$t\langle s_{[0,1]}^{\alpha-1}, k_{\alpha-1}(st)\widetilde{P_\alpha f}(st)\rangle =$$

$$= \langle k_{\alpha-1}(st)s_{[0,1]}^{\alpha-1}, s\frac{\partial^2}{\partial s^2}\tilde{f}(st) + \alpha\frac{\partial}{\partial s}\tilde{f}(st) - t|\xi|^2\tilde{f}(st)\rangle$$

$$= t\langle(\tilde{P}_\alpha k_{\alpha-1})(st)s_{[0,1]}^{\alpha-1},\tilde{f}(st)\rangle - t\frac{\partial}{\partial t}k_{\alpha-1}(\xi,t)\tilde{f}(\xi,t) + tk_{\alpha-1}(\xi,t)\frac{\partial}{\partial t}\tilde{f}(\xi,t)$$

and $\tilde{P}_\alpha k_{\alpha-1} = 0$.

Substitution and addition gives

$$F_\alpha P_\alpha f = \frac{-\pi}{\sin(1-\alpha)\pi}\left[-(1-\alpha)k_{1-\alpha}k_{\alpha-1} + tk_{1-\alpha}\frac{\partial}{\partial t}k_{\alpha-1} - tk_{\alpha-1}\frac{\partial}{\partial t}k_{1-\alpha}\right]\tilde{f} = \tilde{f}. \quad \square$$

The next proposition constitutes the main part of the proof of Theorem 5.5.11.

PROPOSITION 5.6.2. *Let* α, f *and* F_α *be as in* Proposition 5.6.1. *Let* $T > 0$ *be arbitrary. Then*

$$\forall\, p\colon \forall\, m\colon \exists\, K_{p,m} < \infty\colon$$

$$\forall\, t\colon |t| < T\colon \forall\, \xi\colon |\xi|^m \left|\frac{\partial^p}{\partial t^p}(F_\alpha f)(\xi,t)\right| \le K_{p,m}.$$

Moreover, if (f_j) *is a sequence in* $C_0^\infty(\mathbb{R}^{n+1})$ *for which* $f_j \to 0$ *in* $C_0^\infty(\mathbb{R}^{n+1})$, *then the constants* $K_{p,m}^j := K_{p,m}(f_j)$ *can be chosen such that*

$$\forall\, p\colon \forall\, m\colon K_{p,m}^j \to 0 \quad (j \to \infty).$$

PROOF. $F_\alpha f$ was constructed so that it would be smooth and not exponentially increasing for $t > 0$ fixed, $|\xi| \to \infty$. The analysis of F_α should make use of these two properties.

Let $\chi \in C_0^\infty(\mathbb{R})$ be such that $0 \le \chi \le 1$ and $\chi(y) = \begin{cases} 1 & \text{for } |y| \le 1 \\ 0 & |y| \ge 4 \end{cases}$.

Then $\chi_1(\xi,t) := \chi(4t|\xi|^2)$ is a C^∞-function such that

$$\chi_1(\xi,t) = \begin{cases} 1 & \text{for } |t||\xi|^2 \le \frac{1}{4} \\ 0 & \phantom{\text{for }} |t||\xi|^2 \ge 1. \end{cases}$$

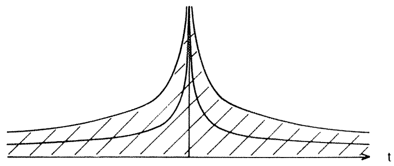

Fig. 20: support of χ_1.

Further $\chi_2(\xi,t) := \mathbb{1} - \chi_1(\xi,t)$.

For $j = 1,2$ and $\ell = 1,2$ we now define $F_\alpha^{(j,\ell)}f$ by

$$(F_\alpha^{(j,\ell)}f)(\xi,t) := \frac{-\pi}{\sin(1-\alpha)\pi} \times$$
$$\Big\{\psi(\xi)(2|\xi|)^{2(\alpha-1)}(\chi_j k_{\alpha-1})(\xi,t) \langle s_+^{\alpha-1}, (\chi_\ell k_{\alpha-1}\tilde{f})(\xi,s)\rangle$$
$$-(\chi_j k_{\alpha-1})(\xi,t) \int_t^\infty (\chi_\ell k_{1-\alpha}\tilde{f})(\xi,s)ds$$
$$-t(\chi_j k_{1-\alpha})(\xi,t) \langle s_{[0,1]}^{\alpha-1}, (\chi_\ell k_{\alpha-1}\tilde{f})(\xi,st)\rangle\Big\}.$$

Then each $F_\alpha^{(j,\ell)}f$ is welldefined and smooth and

$$F_\alpha f = \sum_{j=1}^{2} \sum_{\ell=1}^{2} F_\alpha^{(j,\ell)}f$$

because

$$1 = [\chi_1(\xi,t) + \chi_2(\xi,t)][\chi_1(\xi,s) + \chi_2(\xi,s)] = \sum_{j=1}^{2} \sum_{\ell=1}^{2} \chi_j(\xi,t)\chi_\ell(\xi,s).$$

For each (j,ℓ) $F_\alpha^{(j,\ell)}$ will now be shown to satisfy this proposition.
Because $f \in C_0^\infty$ we know that

$$(5.6.3) \qquad \forall\, p: \forall\, m: \exists\, C_{p,m}: \left|\frac{\partial^p\tilde{f}}{\partial t^p}(\xi,t)\right| \le C_{p,m}(1 + |\xi|)^{-m} \quad \forall\, (\xi,t).$$

Let $R > 0$ be such that $f(x,s) = 0$ for $|s| \ge R$ and assume $|t| < T$.

<u>Analysis of $F_\alpha^{(1,1)}f$.</u>

We have $k_\beta(\xi,t) = j_\beta(2i|\xi|t^{\frac{1}{2}})$ and $j_\beta(z)$ is an even, analytic function (see section A.1).
But then $\chi_1(\xi,t)k_\beta(\xi,t) = \varphi(t|\xi|^2)$ with $\varphi(y) \in C^\infty$ zero for $|y| \ge 1$. So

$$(5.6.4) \qquad \forall\, p: \left|\frac{\partial^p}{\partial t^p}(\chi_1 k_\beta)(\xi,t)\right| = \left|\frac{\partial^p}{\partial t^p}\varphi(t|\xi|^2)\right| = |\xi|^{2p}|\varphi^{(p)}(t|\xi|^2)| \le C|\xi|^{2p}.$$

Let $k \in \mathbb{N} \cup \{0\}$ be so that $k+\alpha-1 > 0$. Then

$$<s_+^{\alpha-1}, (\chi_1 k_{\alpha-1}\tilde{f})(\xi,s)> = \left\langle \frac{1}{(\alpha)_k} s_+^{k+\alpha-1}, (-1)^k \frac{\partial^k}{\partial s^k}(\chi_1 k_{\alpha-1}\tilde{f})(\xi,s)\right\rangle$$

$$= \frac{(-1)^k}{(\alpha)_k} \sum_{i=0}^{k} \binom{k}{i} \int_0^{1/|\xi|^2} s^{k+\alpha-1} \frac{\partial^i}{\partial s^i}(\chi_1 k_{\alpha-1}) \frac{\partial^{k-i}}{\partial s^{k-i}} \tilde{f}\, ds.$$

This can be estimated in absolute value for $|\xi| \geq 1$ by

$$C \cdot \max_{0\leq i\leq k} C_{i,m}(1+|\xi|)^{-m} \cdot \frac{1}{|\xi|^2} \cdot \left(\frac{1}{|\xi|^2}\right)^{k+\alpha-1} |\xi|^{2k}.$$

So

$$\left|\frac{\partial^p}{\partial t^p}\psi(\xi)(2|\xi|)^{2(\alpha-1)}(\chi_1 k_{\alpha-1})(\xi,t) <s_+^{\alpha-1}, (\chi_1 k_{\alpha-1}\tilde{f})(\xi,s)>\right| \leq$$

$$C \cdot \max_{0\leq i\leq k} C_{i,m}(1+|\xi|)^{-m+2p-2}, \quad m \text{ arbitrary, } C,k \text{ independent of } f.$$

The next lemma will be used frequently. The proof is obvious.

<u>LEMMA</u>. *Let* $a_1(\xi,t)$ *and* $a_2(\xi,t)$ *be smooth functions such that*

$$\left|\frac{\partial^p}{\partial t^p} a_i(\xi,t)\right| \leq C_i(1+|\xi|)^{\alpha_i+2p} \qquad \forall\ t,\xi,p,i.$$

Then $\left|\frac{\partial^p}{\partial t^p}(a_1 a_2)(\xi,t)\right| \leq C(1+|\xi|)^{\alpha_1+\alpha_2+2p} \quad \forall\ t,\xi,p.$ □

Now

$$(5.6.5) \qquad \left|\int_t^\infty (\chi_1 k_{1-\alpha}\tilde{f})(\xi,s)ds\right| \leq \int_t^\infty |\chi_1 k_{1-\alpha}\tilde{f}|ds \leq$$

$$\int_{|s|\leq\min\{R,1/|\xi|^2\}} |\chi_1 k_{1-\alpha}\tilde{f}|ds \leq C\cdot C_{0,m}(1+|\xi|)^{-m-2},$$

so

$$\left|\frac{\partial^p}{\partial t^p}(\chi_1 k_{\alpha-1}) \int_t^\infty (\chi_1 k_{1-\alpha}\tilde{f})ds\right| \leq C \cdot \max_{0\leq i\leq p} C_{i,m}(1+|\xi|)^{-m+2p-2},$$

m arbitrary, C independent of f.

Finally:

$$(5.6.6) \qquad \frac{\partial^p}{\partial t^p}<s_{[0,1]}^{\alpha-1}, (\chi_1 k_{\alpha-1}\tilde{f})(\xi,st)> = <s_{[0,1]}^{p+\alpha-1}, \left[\frac{\partial^p}{\partial s^p}(\chi_1 k_{\alpha-1}\tilde{f})\right](\xi,st)>$$

$$= \frac{1}{(\alpha+p)_k}\left[<s_{[0,1]}^{\alpha-1+p+k}, (-1)^k t^k \left[\frac{\partial^{p+k}}{\partial s^{p+k}}(\chi_1 k_{\alpha-1}\tilde{f})\right](\xi,st)>\right.$$

$$\left. + \sum_{i=0}^{k-1} (\alpha+p+i+1)_{k-i-1}(-1)^i t^i \left[\frac{\partial^{p+i}}{\partial t^{p+i}}(\chi_1 k_{\alpha-1}\tilde{f})\right](\xi,t)\right].$$

If $\alpha-1+p+k \geq 0$, this can be estimated by

$$C\cdot \max_{0\leq i\leq p+k} C_{i,m} \sum_{i=0}^{k} |t|^i |\xi|^{2(p+i)}(1+|\xi|)^{-m} \leq C\cdot \max_{0\leq i\leq p+k} C_{i,m}(1+|\xi|)^{-m+2p}.$$

Therefore:

$$\left|\frac{\partial^p}{\partial t^p} t(\chi_1 k_{1-\alpha})(\xi,t) < s^{\alpha-1}_{[0,1]}, (\chi_1 k_{\alpha-1}\tilde{f})(\xi,st)>\right|$$

$$\leq C\cdot \max_{0\leq i\leq p+k} C_{i,m}\left[|t|(1+|\xi|)^{-m+2p} + p(1+|\xi|)^{-m+2(p-1)}\right]$$

$$\leq C\cdot \max_{0\leq i\leq p+k} C_{i,m}(1+|\xi|)^{-m+2p-2} \text{ for } |t| < T, \text{ C,k independent of f.}$$

Conclusion:

$$\forall\ m,p:\ \exists\ K_{m,p}:\ \forall\ (\xi,t):\ |t| < T:$$

$$\left|\frac{\partial^p}{\partial t^p} F^{(1,1)}_\alpha f\right| \leq K_{m,p}\cdot \max_{0\leq i\leq p+k} C_{i,m}(1+|\xi|)^{-m+2p-2}.$$

Here $K_{m,p}$ is independent of f and $k \in \mathbb{N} \cup \{0\}$ should be chosen so that $k \geq 1-\alpha$.

Analysis of $F^{(2,2)}_\alpha f$.

The cut-off functions allow us to rewrite $F^{(2.2)}_\alpha f$:

$$F^{(2,2)}_\alpha f = \frac{-\pi}{\sin(1-\alpha)\pi} \times$$

$$\Big\{(\chi_2 k_{\alpha-1})(\xi,t) \int_t^\infty [\psi(\xi)(2|\xi|)^{2(\alpha-1)} s^{\alpha-1} k_{\alpha-1} - k_{1-\alpha}](\chi_2\tilde{f})(\xi,s)ds$$

$$+\chi_2[\psi(\xi)(2|\xi|)^{2(\alpha-1)} k_{\alpha-1} - t^{1-\alpha} k_{1-\alpha}](\xi,t) \int_0^t s^{\alpha-1}(\chi_2 k_{\alpha-1}\tilde{f})(\xi,s)ds\Big\}.$$

For $|\xi| \geq 1$, $t \neq 0$:

$$\psi(\xi)(2|\xi|)^{2(\alpha-1)} k_{\alpha-1} - t^{1-\alpha} k_{1-\alpha} =$$

$$= i \sin(1-\alpha)\pi e^{(1-\alpha)\pi i}(2i|\xi|)^{\alpha-1} t^{\frac{1}{2}(1-\alpha)} H^{(1)}_{1-\alpha}(2i|\xi|t^{\frac{1}{2}}),$$

$$k_{\alpha-1}(\xi,t) = (2i|\xi|t^{\frac{1}{2}})^{1-\alpha} J_{\alpha-1}(2i|\xi|t^{\frac{1}{2}}) =$$

$$= \tfrac{1}{2}(2i|\xi|)^{1-\alpha} t^{\frac{1}{2}(1-\alpha)}[H^{(1)}_{\alpha-1}(2i|\xi|t^{\frac{1}{2}}) + H^{(2)}_{\alpha-1}(2i|\xi|t^{\frac{1}{2}})].$$

Lemma A.4.1 shows that after multiplication by $\chi_2(\xi,t)$ these expressions can be written as

$$i \sin(1-\alpha)\pi e^{(1-\alpha)\pi i}(2i|\xi|)^{\alpha-1} t^{\frac{1}{2}(1-\alpha)} e^{-2|\xi|t^{\frac{1}{2}}} \ell^+_{1-\alpha}(\xi,t)$$

and

$$\tfrac{1}{2}(2i|\xi|)^{1-\alpha}t^{\frac{1}{2}(1-\alpha)} \times$$

$$\left[e^{-2|\xi|t^{\frac{1}{2}}}\ell^{+}_{\alpha-1}(\xi,t)(1-e^{(\alpha-1)\pi i}2\cos(\alpha-1)\pi) + e^{2|\xi|t^{\frac{1}{2}}}\ell^{-}_{\alpha-1}(\xi,t)\right].$$

Factors t^{β} we estimate as follows:
We can assume $|t|^{\frac{1}{2}}|\xi| \geq \frac{1}{2}$ and $|t| < T$ so

$$\left|\frac{\partial^p}{\partial t^p}t^{\frac{1}{2}\beta}\right| \leq \begin{cases} C_1 & \\ C_2|\xi|^{2p-\beta} & \end{cases} \text{if } \beta-2p \begin{matrix} > 0 \\ < 0 \end{matrix} \Bigg\} \leq C_3|\xi|^{2p} \text{ for } |\xi| \geq 1 \text{ if } \beta > 0.$$

Now we can estimate $(\partial^p/\partial t^p)F^{(2,2)}_{\alpha}f$ using the estimates of Corollary A.4.6.

$$\left|\int_t^{\infty}[\psi(\xi)(2|\xi|)^{2(\alpha-1)}s^{\alpha-1}k_{\alpha-1} - k_{1-\alpha}](\chi_2\tilde{f})(\xi,s)ds\right|$$

$$\leq C\int_{\max\{t,-R\}}^{R}|s|^{\frac{1}{2}(\alpha-1)}|\xi|^{\alpha-1}\,|e^{-2|\xi|s^{\frac{1}{2}}}\ell^{+}_{1-\alpha}(\xi,s)\tilde{f}(\xi,s)|ds \leq$$

$$\leq \begin{cases} C_1|e^{-2|\xi|t^{\frac{1}{2}}}|C_{0,m}(1+|\xi|)^{-m}, & \alpha < 1 \\ C_2|e^{-2|\xi|t^{\frac{1}{2}}}||\xi|^{\alpha-1}C_{0,m}(1+|\xi|)^{-m}, & \alpha > 1 \end{cases}, \quad |\xi| \geq 1, \ |t| < T.$$

Note that $|e^{-2|\xi|t^{\frac{1}{2}}}| = 1$ for $t < 0$. Further for $p \geq 1$:

(5.6.7)
$$\left|\frac{\partial^p}{\partial t^p}\int_t^{\infty}[\psi(\xi)(2|\xi|)^{2(\alpha-1)}s^{\alpha-1}k_{\alpha-1} - k_{1-\alpha}](\chi_2\tilde{f})(\xi,s)ds\right|$$

$$= \left|\frac{\partial^{p-1}}{\partial t^{p-1}}[\psi(\xi)(2|\xi|)^{2(\alpha-1)}t^{\alpha-1}k_{\alpha-1} - k_{1-\alpha}](\chi_2\tilde{f})(\xi,t)\right|$$

$$\leq \begin{cases} C_3|e^{-2|\xi|t^{\frac{1}{2}}}||\xi|^{2p-2}\max\limits_{0\leq i\leq p-1}C_{i,m}(1+|\xi|)^{-m}, & \alpha < 1 \\ C_4|e^{-2|\xi|t^{\frac{1}{2}}}||\xi|^{2p-2+\alpha-1}\max\limits_{0\leq i\leq p-1}C_{i,m}(1+|\xi|)^{-m}, & \alpha > 1 \end{cases}, \quad \begin{matrix} |\xi| \geq 1, \\ |t| < T. \end{matrix}$$

For $p \geq 0$:

(5.6.8)
$$\left|\frac{\partial^p}{\partial t^p}(\chi_2 k_{\alpha-1})(\xi,t)\right|$$

$$\leq \begin{cases} C_5|e^{2|\xi|t^{\frac{1}{2}}}||\xi|^{2p+1-\alpha}, & \alpha < 1 \\ C_6|e^{2|\xi|t^{\frac{1}{2}}}||\xi|^{2p}, & \alpha > 1 \end{cases}, \quad |\xi| \geq 1, \ |t| < T.$$

Now integration in the first term of $F^{(2,2)}_{\alpha}f$ is over $t \leq s$, so the products of the exponentials remain bounded. Applying the estimates obtained above and once more Leibniz' rule, we conclude that the p^{th} derivative of the first term can be estimated by

$$C\cdot \max_{0\le i\le p} C_{i,m}(1+|\xi|)^{-m+2p+|1-\alpha|},\ C \text{ independent of } f,\ |t|<T.$$

Next:

$$\left|\int_0^t s^{\alpha-1}(\chi_2 k_{\alpha-1}\tilde{f})(\xi,s)ds\right|$$

$$\le C_1 \operatorname{sign}(t)\int_0^t |s|^{\frac{1}{2}(\alpha-1)}|\xi|^{1-\alpha}\left|e^{-2|\xi|s^{\frac{1}{2}}}\ell^+_{\alpha-1}(\xi,s)+e^{2|\xi|s^{\frac{1}{2}}}\ell^-_{\alpha-1}(\xi,s)\right||\tilde{f}(\xi,s)|ds$$

$$\le \begin{cases} C_2 T|e^{2|\xi|t^{\frac{1}{2}}}||\xi|^{2(1-\alpha)}C_{0,m}(1+|\xi|)^{-m}, & \alpha<1 \\ C_3 T|e^{2|\xi|t^{\frac{1}{2}}}||\xi|^{1-\alpha}C_{0,m}(1+|\xi|)^{-m}, & \alpha>1 \end{cases},\quad |\xi|\ge 1,\ |t|<T.$$

In a way similar to the one above we can now derive that the p th derivative of the second term can be estimated by

$$C\cdot \max_{0\le i\le p} C_{i,m}(1+|\xi|)^{-m+2p+|1-\alpha|},\ C \text{ independent of } f,\ |t|<T.$$

This concludes the analysis of $F_\alpha^{(2,2)}f$.

Analysis of $F_\alpha^{(1,2)}f$.

We can rewrite $F_\alpha^{(1,2)}f$ as follows:

$$F_\alpha^{(1,2)}f = \frac{-\pi}{\sin(1-\alpha)\pi}\times$$

$$\Big\{(\chi_1 k_{\alpha-1})(\xi,t)\int_t^\infty [\psi(\xi)(2|\xi|)^{2(\alpha-1)}s^{\alpha-1}k_{\alpha-1}-k_{1-\alpha}](\chi_2\tilde{f})(\xi,s)ds$$

$$+\psi(\xi)(2|\xi|)^{2(\alpha-1)}(\chi_1 k_{\alpha-1})(\xi,t)\int_0^t s^{\alpha-1}(\chi_2 k_{\alpha-1}\tilde{f})(\xi,s)ds$$

$$-t(\chi_1 k_{1-\alpha})(\xi,t)\int_0^1 s^{\alpha-1}(\chi_2 k_{\alpha-1}\tilde{f})(\xi,st)ds\Big\}.$$

We may assume $|t||\xi|^2\le 1$, $|s||\xi|^2\ge \frac{1}{4}$.
The $\int_t^\infty$-integral in the first term was estimated before (see the analysis of $F_\alpha^{(2,2)}f$). However, for $p\ge 1$ its p th derivative is given by

$$\frac{\partial^{p-1}}{\partial t^{p-1}}\left[-\left(\psi(\xi)(2|\xi|)^{2(\alpha-1)}t^{\alpha-1}k_{\alpha-1}(\xi,t)-k_{1-\alpha}(\xi,t)\right)(\chi_2\tilde{f})(\xi,t)\right].$$

Here we may assume $\frac{1}{4}\le |t||\xi|^2\le 1$ so here $|(\partial^p/\partial t^p)t^\beta|\le C|\xi|^{2p-2\beta}$ for all β. If we note that $|e^{-2|\xi|t^{\frac{1}{2}}}|\le 1$ for t real, we may conclude, combining the estimates (5.6.4) and (5.6.7), that the p th derivative of the first term can be estimated by

$$C\cdot \max_{0\le i\le p} C_{i,m}(1+|\xi|)^{-m+2p+|1-\alpha|},\ C \text{ independent of } f,\ |t|<T.$$

In the second term we note that $|e^{2s^{\frac{1}{2}}|\xi|}| \le e^2$ and $|e^{2t^{\frac{1}{2}}|\xi|}| \le e^2$, because for $t > 0$: $s^{\frac{1}{2}}|\xi| \le t^{\frac{1}{2}}|\xi| \le 1$. But then the second term satisfies similar estimates as the first term.
In the third term we note that

$$\frac{\partial^p}{\partial t^p}\int_0^1 s^{\alpha-1}(\chi_2 k_{\alpha-1}\tilde{f})(\xi,st)ds = \int_0^1 s^{p+\alpha-1}\left[\frac{\partial^p}{\partial s^p}(\chi_2 k_{\alpha-1}\tilde{f})\right](\xi,st)ds.$$

$|st|^{\frac{1}{2}}|\xi| \le |t|^{\frac{1}{2}}|\xi| \le 1$ so exponentials are bounded. $|st||\xi|^2 \ge \frac{1}{4}$ and $|t||\xi|^2 \le 1$ so $s \ge \frac{1}{4}$.

By now it should be obvious that the third term satisfies similar estimates as the first term (even with exponent $-m+2p-2+|1-\alpha|$). This concludes the analysis of $F_\alpha^{(1,2)}f$.

Analysis of $F_\alpha^{(2,1)}f$.

We can assume $|t||\xi|^2 \ge \frac{1}{4}$ and $|s||\xi|^2 \le 1$. For $t \le 0$ we have $|e^{\pm 2|\xi|t^{\frac{1}{2}}}| = 1$ so for $t \le 0$ $(\partial^p/\partial t^p)(\chi_2 k_{\alpha-1})(\xi,t)$ can be estimated by

$$C|\xi|^{2p+|1-\alpha|}, \quad |\xi| \ge 1, \quad -T < t \le 0.$$

So for $t \le 0$ $F_\alpha^{(2,1)}f$ can be analysed similarly to $F_\alpha^{(1,1)}f$, giving for the p th derivative the estimate

$$C\cdot\max_{0\le i\le p+k} C_{i,m}(1+|\xi|)^{-m+2p-2+|1-\alpha|}, \text{ C independent of f, } -T<t\le 0, \; k > 1-\alpha.$$

For $t \ge 0$ we rewrite $F_\alpha^{(2,1)}f$ as follows:

$$F_\alpha^{(2,1)}f = \frac{-\pi}{\sin(1-\alpha)\pi} \times$$

$$\Big\{t\chi_2(\xi,t)[\psi(\xi)(2|\xi|)^{2(\alpha-1)}t^{\alpha-1}k_{\alpha-1}-k_{1-\alpha}]<s_{[0,1]}^{\alpha-1},(\chi_1 k_{\alpha-1}\tilde{f})(\xi,st)>$$
$$+t^\alpha\chi_2(\xi,t)\psi(\xi)(2|\xi|)^{2(\alpha-1)}k_{\alpha-1}(\xi,t)<s_{[1,\infty)}^{\alpha-1},(\chi_1 k_{\alpha-1}\tilde{f})(\xi,st)>$$
$$-(\chi_2 k_{\alpha-1})(\xi,t)\int_t^\infty(\chi_1 k_{1-\alpha}\tilde{f})(\xi,s)ds\Big\}.$$

This is obvious for $\alpha > 1$. For $\alpha \in \mathbb{C}\setminus\mathbb{Z}$ it follows by analytic continuation. Note that for ξ fixed $\chi_2(\xi,t) = 0$ in a neighbourhood of $t = 0$. So the second term is welldefined and smooth for $t \ge 0$, too.

The first term consists of factors which have been analysed before (see estimates (5.6.6) and (5.6.7)). Its p th derivatives can be estimated by

$$C \cdot \max_{0 \le i \le p+k} C_{i,m}(1+|\xi|)^{-m+2p+|1-\alpha|}, \text{ C independent of f, } 0 \le t < T, \ |\xi| \ge 1, \ k > 1-\alpha.$$

The third term also consists of factors analysed before (see estimates (5.6.5) and (5.6.8)). We have $t^{\frac{1}{2}}|\xi| \le s^{\frac{1}{2}}|\xi| \le 1$, so exponentials are bounded and we can estimate the derivatives as above (with $k = 0$).

As for the second term, because $t|\xi|^2 \ge \frac{1}{4}$

$$\left|\frac{\partial^p}{\partial t^p} < s^{\alpha-1}_{[1,\infty)}, (\chi_1 k_{\alpha-1}\widetilde{f})(\xi,st)>\right| = \left|\int_1^\infty s^{p+\alpha-1}\left[\frac{\partial^p}{\partial s^p}(\chi_1 k_{\alpha-1}\widetilde{f})\right](\xi,st)ds\right|.$$

Also $|st||\xi|^2 \le 1$, so $|s| \le 1/(t|\xi|^2) \le 4$ and we can estimate this by

$$C \cdot \max_{0 \le i \le p} C_{i,m}|\xi|^{2p}(1+|\xi|)^{-m}, \quad |\xi| \ge 1.$$

Further

$$\left|\frac{\partial^p}{\partial t^p} t^{\alpha-1}(\chi_2 k_{\alpha-1})(\xi,t)\right| \le C\begin{Bmatrix} |\xi|^{2p+2(1-\alpha)} \\ |\xi|^{2p+1-\alpha} \end{Bmatrix} e^{2t^{\frac{1}{2}}|\xi|}, \quad \begin{matrix} \alpha < 1 \\ \alpha > 1 \end{matrix}, \quad |\xi| \ge 1, \ |t| < T.$$

Now $t^{\frac{1}{2}}|\xi| \le (st)^{\frac{1}{2}}|\xi| \le 1$ so exponentials are bounded, so the derivatives of the second term have estimates as those of the third term. This concludes the analysis of $F_\alpha^{(2,1)}f$.

Conclusion: Let $T > 0$ and $\text{supp}(f) \subset \{(x,t) \mid |t| \le R\}$, then

$$\forall p: \forall m: \exists \widetilde{K}_{p,m}: \forall (\xi,t) \text{ with } |t| < T:$$

$$\left|\frac{\partial^p}{\partial t^p}(F_\alpha f)(\xi,t)\right| \le \widetilde{K}_{p,m} \cdot \max_{0 \le i \le p+k} C_{i,m}(1+|\xi|)^{-m+2p+|1-\alpha|}.$$

Here $C_{i,m} = C_{i,m}(f)$, $k \in \mathbb{N} \cup \{0\}$ should be chosen so that $k \ge 1-\alpha$ and $\widetilde{K}_{p,m}$ is depending on T,R,p,m only. In particular, $\widetilde{K}_{p,m}$ is independent of f such that $\text{supp}(f) \subset \{(x,t) \mid |x|^2 + t^2 \le R^2\}$. So $F_\alpha f$ satisfies an estimate as mentioned in the proposition. Let now (f_j) be a sequence in $C_0^\infty(\mathbb{R}^{n+1})$ such that $f_j \to 0$ in $C_0^\infty(\mathbb{R}^{n+1})$, $j \to \infty$. Then $\exists R: \forall j$: $\text{supp } f_j \subset \{(x,t) \mid |x|^2 + t^2 \le R^2\}$ and for $\widetilde{C}^j_{p,\alpha} := \sup_{(x,t)} |(D_x^\alpha D_t^p f_j)(x,t)|$ we have $\forall (p,\alpha)$: $\widetilde{C}^j_{p,\alpha} \to 0$ $(j \to \infty)$. But then

$$|\xi^\alpha D_t^p \widetilde{f}_j(\xi,t)| = \left|\int e^{-i<x,\xi>} D_x^\alpha D_t^p f_j(x,t)dx\right| \le C_R \cdot \widetilde{C}^j_{p,\alpha}$$

with C_R only depending on R.

Now

$$|(1+|\xi|)^m D_t^p \widetilde{f}_j(\xi,t)| \le \sum_{|\alpha| \le m} C_\alpha |\xi^\alpha D_t^p \widetilde{f}_j(\xi,t)| \le \sum_{|\alpha| \le m} C_\alpha C_R \widetilde{C}^j_{p,\alpha}$$

$$\leq C_{m,R} \sum_{|\alpha|\leq m} \tilde{C}^j_{p,\alpha} \quad \text{with } C_{m,R} \text{ depending on } (m,R) \text{ only.}$$

Let $C^j_{p,m} := \sup_{(\xi,t)} |(1+|\xi|)^m D^p_t \tilde{f}_j(\xi,t)|$. Then these $C^j_{p,m}$ establish estimate (5.6.3) for f_j and for fixed (p,m) $C^j_{p,m} \to 0$. But then the constants

$$K^j_{p,m} := \tilde{K}_{p,m} \cdot \max_{0\leq i\leq p+k} C^j_{i,m}$$

satisfy $K^j_{p,m} \to 0$ $(j \to \infty)$, too.
This concludes the proof of Proposition 5.6.2. □

The proof of Theorem 5.5.11. is now an easy matter.

PROOF of Theorem 5.5.11. The Propositions 5.6.1 and 5.6.2 show that $F_\alpha f$ is smooth and

$$|D^\alpha_x D^p_t(e^{+i<x,\xi>}(F_\alpha f)(\xi,t))| = |\xi^\alpha D^p_t(F_\alpha f)(\xi,t)|$$

$$\leq C_{p,\alpha}(1+|\xi|)^{-(n+1)} \quad \text{for } |t| < T, \quad T < \infty \text{ arbitrary.}$$

Therefore $E_\alpha f$ is welldefined and smooth in (x,t). If (f_j) is a sequence in $C^\infty_0(\mathbb{R}^{n+1})$ so that $f_j \to 0$ in $C^\infty_0(\mathbb{R}^{n+1})$ $(j \to \infty)$, then the constants $C_{p,\alpha} = C^j_{p,\alpha}$ can be chosen so that $C^j_{p,\alpha} \to 0$ $(j \to \infty)$. But then for $|t| < T$

$$|D^\alpha_x D^p_t(E_\alpha f_j)(x,t)| = \frac{1}{(2\pi)^n}\left|\int e^{i<x,\xi>}\xi^\alpha D^p_t(F_\alpha f_j)(\xi,t)\,d\xi\right|$$

$$\leq \frac{1}{(2\pi)^n} C^j_{p,\alpha} \int \frac{d\xi}{(1+|\xi|)^{n+1}} \to 0 \quad \text{for } j \to \infty.$$

So $E_\alpha f_j \to 0$ in C^∞. Now obviously E_α is linear so E_α is a continuous map from $C^\infty_0(\mathbb{R}^{n+1})$ to $C^\infty(\mathbb{R}^{n+1})$. Finally

$$P_\alpha E_\alpha f = (\tilde{P}_\alpha \widetilde{E_\alpha f})\check{} = (\tilde{P}_\alpha F_\alpha f)\check{} = (\tilde{f})\check{} = f\,,$$

$$E_\alpha P_\alpha f = (F_\alpha P_\alpha f)\check{} = (\tilde{f})\check{} = f\,,$$

as follows from Proposition 5.6.1. □

5.7. Qualitative properties of ${}^{(t)}E_\alpha$.

In this section we will discuss the way ${}^tE_\alpha$ propagates singularities. In the first place we remark that from formula (5.5.9) it follows that the kernel of E_α has support contained in $\{(x,t,y,s) \mid s \leq 0 \Rightarrow t \leq s\}$. So the kernel of ${}^tE_\alpha$ has support contained in $\{(x,t,y,s) \mid t \leq 0 \Rightarrow s \leq t\}$. Note that this implies that we avoid the set C_2 given in Proposition 5.2.5. See also paragraph (5.2.6). Therefore, if $\operatorname{supp}(f) \subset \{(y,s) \mid s \geq -M\}$ for some $M \geq 0$ then $\operatorname{supp}({}^tE_\alpha f) \subset \{(x,t) \mid t \geq -M\}$. In particular we obtain

PROPOSITION 5.7.1. *Let* $f \in \mathcal{E}'(\mathbb{R}^{n+1})$. *Then* $\mathrm{WF}({}^tE_\alpha f|_{t<0})$ *does not contain a complete bicharacteristic strip of* P_α.

PROOF. Along such a strip $t \to -\infty$ (see section 5.2). □

From Proposition 5.4.5 we learn that for some $f \in C_0^\infty(\mathbb{R}^{n+1})$ ${}^tE_\alpha f$ cannot be smooth. Together with Proposition 5.7.1 and the fact that $\mathrm{WF}({}^tE_\alpha f) \subset N = \{(x,t,\xi,\tau) \mid t\tau^2 + |\xi|^2 = 0\}$ this shows that for such f:

$$(5.7.2) \qquad \emptyset \neq \mathrm{WF}({}^tE_\alpha f) \subset \{(x,0,0,\tau) \mid \tau \neq 0\}.$$

Of course, the set C_1 given in Proposition 5.2.5 is responsible for this fact.

Finally we show that for $f \in \mathcal{E}'(\mathbb{R}^n \times \mathbb{R}^-)$ the singularities of ${}^tE_\alpha f$ for $t < 0$ can be obtained by expressing ${}^tE_\alpha$ in terms of FIOs. This should not be too surprising since P_α is of real principal type on $\mathbb{R}^n \times \mathbb{R}^-$ and $\mathbb{R}^n \times \mathbb{R}^-$ is pseudo convex with respect to P_α (see section 5.2).

For $f \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^-)$ the restriction of $E_\alpha f$ to $t < 0$ is given by

$$\frac{1}{(2\pi)^n} \int d\xi e^{i\langle x,\xi\rangle} \frac{-\pi}{\sin(1-\alpha)\pi} \times$$

$$\int_t^\infty ds[-k_{\alpha-1}(\xi,t)k_{1-\alpha}(\xi,s) + t^{1-\alpha}k_{1-\alpha}(\xi,t)s^{\alpha-1}k_{\alpha-1}(\xi,s)]\tilde{f}(\xi,s).$$

So the restriction of the kernel of E_α to $s < 0$ and $t < 0$ is given by

$$H(s-t)\frac{1}{(2\pi)^n} \int d\xi e^{i\langle x-y,\xi\rangle} e_\alpha(t,s,\xi)$$

with

$$e_\alpha(t,s,\xi) =$$

$$= \frac{-\pi}{\sin(1-\alpha)\pi}[-k_{\alpha-1}(\xi,t)k_{1-\alpha}(\xi,s) + t^{1-\alpha}k_{1-\alpha}(\xi,t)s^{\alpha-1}k_{\alpha-1}(\xi,s)]$$

$$= \frac{-\pi}{\sin(1-\alpha)\pi} t^{\frac{1}{2}(1-\alpha)} s^{\frac{1}{2}(\alpha-1)} \times$$

$$[J_{1-\alpha}(2i|\xi|t^{\frac{1}{2}})J_{\alpha-1}(2i|\xi|s^{\frac{1}{2}}) - J_{\alpha-1}(2i|\xi|t^{\frac{1}{2}})J_{1-\alpha}(2i|\xi|s^{\frac{1}{2}})]$$

$$= \frac{-\pi i}{2} t^{\frac{1}{2}(1-\alpha)} s^{\frac{1}{2}(\alpha-1)} \times$$

$$[H^{(2)}_{1-\alpha}(2i|\xi|t^{\frac{1}{2}})H^{(1)}_{1-\alpha}(2i|\xi|s^{\frac{1}{2}}) - H^{(1)}_{1-\alpha}(2i|\xi|t^{\frac{1}{2}})H^{(2)}_{1-\alpha}(2i|\xi|s^{\frac{1}{2}})].$$

See section 5.3 and section A.1. For $\xi \neq 0$ this is equal to

$$\frac{-\pi i}{2} t^{\frac{1}{2}(1-\alpha)} s^{\frac{1}{2}(\alpha-1)} \times$$
$$\Big[h^{-}_{1-\alpha}(t,\xi)h^{+}_{1-\alpha}(s,\xi)e^{2i|\xi|((-s)^{\frac{1}{2}}-(-t)^{\frac{1}{2}})}$$
$$- h^{+}_{1-\alpha}(t,\xi)h^{-}_{1-\alpha}(s,\xi)e^{2i|\xi|((-t)^{\frac{1}{2}}-(-s)^{\frac{1}{2}})}\Big].$$

See section A.4.

As in section 3.4 we introduce a function $\chi \in C^{\infty}$ so that

$$\chi(\xi) = \begin{cases} 1 & \text{for } |\xi| < 1 \\ 0 & \text{for } |\xi| > 2. \end{cases}$$

Then the kernel

$$\int d\xi e^{i<x-y,\xi>} e_\alpha(t,s,\xi)$$

can be written as the sum of a smooth function and two FIOs with phase functions $<x-y,\xi> \pm 2|\xi|((-t)^{\frac{1}{2}} - (-s)^{\frac{1}{2}})$ and symbols $[1-\chi(\xi)]h^{\pm}_{1-\alpha}(t,\xi)h^{\mp}_{1-\alpha}(s,\xi)$ respectively. The phase functions are the functions $\varphi_{\mp}$ given in section 5.2. The symbols are elliptic elements of $S^{-1}_{1,0}$ (see section A.4).

$$\Lambda_{\varphi_{\mp}} = \{(y \mp 2\frac{\xi}{|\xi|}((-t)^{\frac{1}{2}}-(-s)^{\frac{1}{2}}),t,\xi,\mp(-t)^{-\frac{1}{2}}|\xi|;y,s,-\xi,\pm(-s)^{-\frac{1}{2}}|\xi|)\,\Big|\, t<0, s<0, \xi \neq 0\}.$$

As in section 3.4 we see that multiplication with H(s-t) is welldefined and that the wave front set of the kernel of E_α restricted to $s < 0$ and $t < 0$ is contained in

$$\text{(5.7.3)} \qquad WF(H(s-t)) \cup (\Lambda_{\varphi_+} \cup \Lambda_{\varphi_-})\big|_{s\geq t} \cup \{(x,t,\xi,\tau;x,t,-\xi,-\tau) \mid t < 0\}.$$

Now the kernel of ${}^tE_\alpha$ is obtained from the kernel of E_α by exchanging (x,t) and (y,s). But then it is clear from formula (5.7.3) that indeed a singularity of f can only be propagated by ${}^tE_\alpha$ along the strip through that point and only in one direction.

Moreover we have

<u>COROLLARY 5.7.4.</u> *Let* $f \in \mathcal{E}'$ *be singular in some point* (x_0,t_0,ξ_0,τ_0) *of* N *with* $t_0 < 0$. *Suppose* f *is smooth in points* (y,s,η,σ) *on the strip through that point for* $s < t_0$. *Then* ${}^tE_\alpha f$ *is singular in points* (x,t,ξ,τ) *on this*

strip for $t > t_0$ *until the strip hits another point in* WF(f).

PROOF. The symbols are elliptic. □

Remark 5.7.5. Property (5.7.2) shows that it is senseless to try to describe ${}^{t}E_{\alpha}f$ in terms of FIOs in a neighbourhood of $t = 0$. For a FIO has a phase function satisfying the conditions given in paragraph (2.8.6), so it maps C_0^∞ to C^∞.

5.8. Smooth solutions of $P_\alpha u = 0$.

In this section we have a closer look at the solutions of the homogeneous equation. In particular we are interested in smooth solutions. The results we obtain are not complete but we can determine all smooth solutions of $P_\alpha u = 0$ for $t < 0$ which are smooth even up to $t = 0$. We only have partial results for smooth solutions of $P_\alpha u = 0$ for $t > 0$.

If $t_0 < 0$ and $P_\alpha u = 0$ then $f := u\big|_{t=t_0}$ and $g := \frac{\partial u}{\partial t}\big|_{t=t_0}$ determine u completely for $t < 0$ since P_α is hyperbolic for $t < 0$ and $\{(x,t) \mid t = t_0\}$ is nowhere characteristic. Therefore we first consider for $t_0 < 0$ the problem:

$$(5.8.1)\qquad \begin{cases} P_\alpha u = 0 \quad \text{for } t < 0,\ x \in \mathbb{R}^n, \\ u\big|_{t=t_0} = f, \\ \frac{\partial u}{\partial t}\big|_{t=t_0} = g. \end{cases}$$

Again we proceed formally in order to obtain a formula for the solution. Partial Fourier transformation with respect to x gives

$$\tilde{P}_\alpha \tilde{u} = (t\frac{\partial^2}{\partial t^2} + \alpha\frac{\partial}{\partial t} - |\xi|^2)\tilde{u} = 0, \quad \tilde{u}\big|_{t=t_0} = \hat{f}, \quad \frac{\partial \tilde{u}}{\partial t}\big|_{t=t_0} = \hat{g}.$$

Solutions of $\tilde{P}_\alpha \tilde{u} = 0$ are

$$\tilde{u}(\xi,t) = c_1(\xi)t^{\frac{1}{2}(1-\alpha)}J_{1-\alpha}(2i|\xi|t^{\frac{1}{2}}) + c_2(\xi)t^{\frac{1}{2}(1-\alpha)}J_{\alpha-1}(2i|\xi|t^{\frac{1}{2}}).$$

See section 5.3. Here for $t < 0$, $t^\beta = \exp \beta(\log(-t)+\pi i)$. Substituting the boundary conditions for $t = t_0$, c_1 and c_2 can be expresses in terms of f and g by

$$\begin{pmatrix} c_1(\xi) \\ c_2(\xi) \end{pmatrix} = \frac{\pi|\xi|\sqrt{-t_0}}{\sin(1-\alpha)\pi}\, t_0^{\frac{1}{2}(\alpha-1)} \times$$

$$\begin{pmatrix} -\frac{\sqrt{-t_0}}{|\xi|} J_{\alpha-1}(z)\hat{g}(\xi) + [J'_{\alpha-1}(z) - \frac{1-\alpha}{2|\xi|\sqrt{-t_0}} J_{\alpha-1}(z)]\hat{f}(\xi) \\ \frac{\sqrt{-t_0}}{|\xi|} J_{1-\alpha}(z)\hat{g}(\xi) - [J'_{1-\alpha}(z) - \frac{1-\alpha}{2|\xi|\sqrt{-t_0}} J_{1-\alpha}(z)]\hat{f}(\xi) \end{pmatrix}.$$

Here $z = 2i|\xi|t_0^{\frac{1}{2}}$.

(5.8.2) Using the relations $zJ'_\nu(z) \pm \nu J_\nu(z) = \pm zJ_{\nu\mp 1}(z)$ and the definition $k_\nu(\xi,t) = (2i|\xi|t^{\frac{1}{2}})^{-\nu}J_\nu(2i|\xi|t^{\frac{1}{2}})$ from section 5.3, the solution $\tilde{u}$ can be written as:

$$(5.8.3) \qquad \tilde{u}(\xi,t) = \frac{\pi t_0^\alpha}{\sin(1-\alpha)\pi}[k_{\alpha-1}(\xi,t_0)\hat{g}(\xi)-2|\xi|^2k_\alpha(\xi,t_0)\hat{f}(\xi)]t^{1-\alpha}k_{1-\alpha}(\xi,t)$$

$$+ \frac{\pi}{\sin(1-\alpha)\pi}[-t_0k_{1-\alpha}(\xi,t_0)\hat{g}(\xi) + \tfrac{1}{2}k_{-\alpha}(\xi,t_0)\hat{f}(\xi)]k_{\alpha-1}(\xi,t).$$

Note that but for $\hat{f}$ and $\hat{g}$, the only singular factors in this expression are t_0^α and $t^{1-\alpha}$ for $t_0 \uparrow 0$ and $t \uparrow 0$.

LEMMA 5.8.4.

1. *For fixed* $t \le 0$, $k_\nu(\xi,t)$ *is the Fourier transform of a distribution with support in* $\{x \mid |x| \le 2\sqrt{-t}\}$.
2. *Let* $K_\nu(x,t)$ *denote this distribution. Then* $K_\nu \in C^\infty(\overline{\mathbb{R}^-},\mathcal{E}'(\mathbb{R}_x))$.
3. *If* $\varphi_j \to 0$ *in* $C^\infty(\mathbb{R}^n_x)$ *then* $\langle K_\nu(x,t),\varphi_j(x)\rangle \to 0$ *in* $C^\infty(\overline{\mathbb{R}^-})$.

PROOF. The analyticity of k_ν follows from Proposition 5.3.4. The asymptotic expansions of the Bessel functions provide the Paley-Wiener estimates. Corollary A.4.6 and estimate (5.6.4) show that for arbitrary $T > 0$

$$(5.8.5) \qquad \forall\, k: \exists\, C_k: \left|\frac{\partial^k}{\partial t^k}k_\nu(\xi,t)\right| \le C_k(1+|\xi|)^{2k}, \qquad -T \le t \le 0.$$

The rest of the proof is now similar to the proof of Lemma 4.2.4. □

It should now be clear that for arbitrary f and g in $C^\infty(\mathbb{R}^n)$, the (unique) solution of problem (5.8.1) can be obtained from formula (5.8.3) by interpreting $[k_{\alpha-1}(\xi,t)\hat{f}(\xi)]^\vee$ as the convolution of $K_{\alpha-1}(x,t)$ and f with respect to x, etc. Again, FIO-representations can be obtained by inserting the asymptotic expansions of the Bessel functions. The phase functions which appear are

$$\langle x-y,\xi\rangle \pm 2|\xi|(\sqrt{-t} - \sqrt{-t_0}).$$

One might also expect the phase functions $\langle x-y,\xi\rangle \pm 2|\xi|(\sqrt{-t} + \sqrt{-t_0})$ to appear. However, the terms involving these functions cancel.

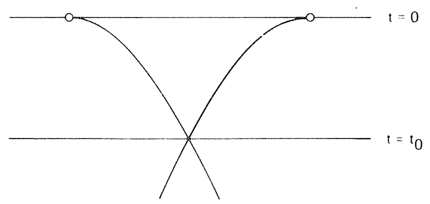

Fig. 21: Cauchy problem on $t = t_0$.

Of course, this expresses the fact that a bicharacteristic strip, unlike the situation in case of the Tricomi operator, does not reflect at $t = 0$ but only approaches asymptotically.

Next we will determine the solutions for $t < 0$ which remain smooth up to $t = 0$.

<u>LEMMA 5.8.6</u>. *Suppose* $f \in C^\infty(\mathbb{R}^n)$. *Then* $K_\nu(x,s) *_x f$ *is smooth for* $s \le 0$.

<u>PROOF</u>. For $f \in C_0^\infty(\mathbb{R}^n)$, $K_\nu(x,s) *_x f = 1/(2\pi)^n \int e^{i<x,\xi>} k_\nu(\xi,s)\hat{f}(\xi)d\xi$. Then $\hat{f}(\xi)$ is rapidly decreasing, $k_\nu(\xi,s)$ is smooth for all (ξ,s), so estimate (5.8.5) shows that $K_\nu(x,s) *_x f$ is smooth for $s \le 0$. For $f \in C^\infty(\mathbb{R}^n)$ the result follows from Lemma 5.8.4, part 1. □

From this Lemma it follows that Fourier's inversion formula applied to the second term on the right in equation (5.8.3) gives a function which is smooth for $t \le 0$, provided $f \in C^\infty$ and $g \in C^\infty$. The same holds for the other term but for the factor $t^{1-\alpha}$. Note that $\alpha \notin \mathbb{Z}$. Since $k_\nu(\xi,0) = j_\nu(0) = 2^{-\nu}(1/\Gamma(\nu+1)) \neq 0$, a necessary condition for this term (including $t^{1-\alpha}$) to be smooth for $t \le 0$ is:

$$K_{\alpha-1}(x,t_0) *_x g + 2\Delta_x K_\alpha(x,t_0) *_x f = 0.$$

This condition is also sufficient. (Note that for $t_0 = 0$ this condition simply states $(P_\alpha u)(x,0) = 0$.) But then substitution of $t_0 = 0$ gives that $u(x,t)$ has the representation

(5.8.7) $$u(x,t) = c_\alpha u(x,0) *_x K_{\alpha-1}(x,t).$$

Here $c_\alpha = 1/j_{\alpha-1}(0) = 2^{\alpha-1}\Gamma(\alpha)$. So we have derived

PROPOSITION 5.8.8. *u is a solution of* $P_\alpha u = 0$ *for* $t < 0$ *which is smooth up to* $t = 0 \Leftrightarrow u(x,t) = f(x) *_x c_\alpha K_{\alpha-1}(x,t)$ *for some* $f \in C^\infty$.
Moreover, in that case we have, with $*$ *convolution with respect to* x *only*:

1) $f(x) = u(x,0)$,

2) $K_{\alpha-1}(x,t) * \frac{\partial u}{\partial t}(x,t) - \frac{\partial K_{\alpha-1}}{\partial t}(x,t) * u(x,t) = 0$ *for* $t \le 0$,

3) $-tK_{1-\alpha}(x,t) * \frac{\partial u}{\partial t}(x,t) + \frac{1}{2}K_{-\alpha}(x,t) * u(x,t) = \frac{u(x,0)}{2^{1-\alpha}\Gamma(1-\alpha)}$, $t \le 0$.

PROOF. Most results follow from the discussion above. In statement 2) we expressed $(|\xi|^2 k_\alpha)^\wedge$ in terms of $\partial K_{\alpha-1}/\partial t$ using the relations (5.8.2). □

REMARK. Of course, the fact that $\{(x,t) \mid t = 0\}$ is characteristic is responsible for the fact that we can only prescribe u on $t = 0$ and not $\frac{\partial u}{\partial t}$. However, we showed that there is a unique u which is smooth up to $t = 0$ and satisfies $P_\alpha u = 0$ for $t < 0$, $u|_{t=0} = f \in C^\infty$. Alternatively, if u is smooth for $t < 0$ and we require u to be C^k, $k < \infty$, up to $t = 0$, then equation (5.8.3) shows similarly that this is the case for arbitrary $f \in C^\infty$ and $g \in C^\infty$ if and only if $1-\alpha > k$. If $1-\alpha < k$, u is forced to be of the form (5.8.7). This fact accounts for the different boundary value problems discussed in Karol [17] for different ranges of α (see section 5.1).

We now discuss smooth solutions for $t > 0$.
In this case our results are less complete. This is caused by the fact that in the elliptic case we do not have representation formulas like formula (5.8.3) for the solution of a Cauchy problem. However we will give a partial result in the spirit of Garabedian.

Because P_α is elliptic for $t > 0$ we know that $P_\alpha u = 0$ implies u is real analytic for $t > 0$. We will assume that $u = u(z,t)$, $z = x + iy$, even is entire in z for $t > 0$ and we will determine all such u which are smooth in (x,t) and (y,t) for $t \ge 0$.

Define for $t < 0$ $v(z,t) := u(iz,-t)$.
Then v is welldefined, entire in z for $t < 0$ and

$$(P_\alpha v)(x,t) = -(P_\alpha u)(ix,-t) = 0.$$

Let moreover

$$f(x) = v(x,t_0) = u(ix,-t_0),$$
$$g(x) = \frac{\partial v}{\partial t}(x,t_0) = -\frac{\partial u}{\partial t}(ix,-t_0), \quad t_0 < 0.$$

Then f and g are entire and v can be expressed in terms of f and g by means of formula (5.8.3). Note that u is smooth in (y,t) for $t \geq 0$ implies $v(x,t)$ is smooth in (x,t) for $t \leq 0$. So v is given for $t \leq 0$ by

$$v(x,t) = c_\alpha v(x,0) *_x K_{\alpha-1}(x,t).$$

We did not use here that u is smooth in (x,t) up to $t = 0$! Proposition 5.8.8, part 3 shows that $v(x,0)$ is entire in x. But then u is given by

$$\begin{aligned} u(x,t) = v(-ix,-t) &= c_\alpha \langle K_{\alpha-1}(\eta,-t), v(-ix-\eta,0)\rangle \\ &= c_\alpha \langle K_{\alpha-1}(\eta,-t), u(x-i\eta,0)\rangle. \end{aligned}$$

We denote this by $u(x,t) = u_0 \tilde{*} c_\alpha K_{\alpha-1}(x,t)$.

<u>PROPOSITION 5.8.9</u>. *u is a solution of* $P_\alpha u = 0$ *for* $t > 0$ *which is entire in* $z = x + iy$ *for* $t > 0$ *and smooth in* (x,t) *and* (y,t) *for* $t \geq 0$ $\Leftrightarrow u(x,t) = f \tilde{*} c_\alpha K_{\alpha-1}(x,t)$ *for some entire function* f.
Moreover, in that case we have $f(x) = u(x,0)$.

<u>PROOF</u>. $\Rightarrow$ follows from the discussion above.
$\Leftarrow$ That u is analytic in $z = x + iy$ for $t \geq 0$ is evident. Furthermore we can prove that $\langle K_{\alpha-1}(\eta,-t), \omega(x,y,\eta)\rangle$ is smooth in (x,y,t) for $t \geq 0$ for arbitrary $\omega \in C^\infty$. We can assume $\omega \in C_0^\infty$ and then the result follows as before from estimate (5.8.5). That $P_\alpha u = 0$ follows from the fact that $\Delta_{(x,\eta)} f(x-i\eta) = 0$.
The last statement is evident. □

We might try to extend the result of Proposition 5.8.9 to smooth functions f so that for some smooth $a = a(x,\eta)$: $\Delta_{(x,\eta)} a = 0$ and $a(x,0) = f(x)$. For bounded f such a function might be given by Poisson's formula for harmonic functions. However, we do not see how to obtain such a for arbitrary smooth f.

<u>COROLLARY 5.8.10</u>. *If* f *is entire then*

$$u(x,t) := \begin{cases} f(x) \tilde{*} c_\alpha K_{\alpha-1}(x,-t), & t \geq 0 \\ f(x) *_x c_\alpha K_{\alpha-1}(x,t), & t \leq 0 \end{cases}$$

defines a smooth solution of $P_\alpha u = 0$, $u|_{t=0} = f$.

<u>PROOF</u>. The smoothness follows from the fact that the expressions defining u are equal on $t = 0$ and satisfy $P_\alpha u = 0$ for $t \geq 0$, $t \leq 0$ respectively. □

APPENDICES

A.1. Bessel functions.

In this section we summarize some wellknown facts about Bessel functions. See Watson [27].

For $n \in \mathbb{Z}$ the Bessel function $J_n(z)$ is defined by

$$J_n(z) = \frac{1}{2\pi i} \oint_{|w|=1} \frac{1}{w^{n+1}} \exp\left(\tfrac{1}{2}z\left(w - \tfrac{1}{w}\right)\right) dw. \tag{A.1.1.}$$

It is clear that $J_n(z)$ is an entire function in z. Moreover $J_{-n}(z) = (-1)^n J_n(z)$.

More generally, for $\nu \in \mathbb{C}$ $J_\nu(z)$ is defined by

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\tfrac{1}{2}z)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)} . \tag{A.1.2}$$

This coincides with definition (A.1.1) when $\nu = 0,1,2,\ldots$ etc. The function

$$j_\nu(z) := z^{-\nu} J_\nu(z) \tag{A.1.3}$$

can be extended to an entire function in z. That is, we can define branches of $J_\nu(z)$ in the same way as we define branches of z^ν.
Note that $j_\nu(-z) = j_\nu(+z)$.

For $\nu \in \mathbb{C}$ $J_\nu(z)$ is a solution of Bessel's equation

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} + (z^2 - \nu^2)u = 0.$$

So is $J_{-\nu}(z)$. For $\nu \notin \mathbb{Z}$ these two functions are linear independent.

More generally, the equation

(A.1.4) $$z^2\frac{d^2u}{dz^2} + (2\alpha - 2\beta\nu + 1)z\frac{du}{dz} + (\beta^2\gamma^2 z^{2\beta} + \alpha(\alpha-2\beta\nu))u = 0$$

is solved by $u = z^{\beta\nu-\alpha} J_{\pm\nu}(\gamma z^{\beta})$.

Some formulas:

(A.1.5) $$\begin{vmatrix} J_\nu(z) & J_{-\nu}(z) \\ J'_\nu(z) & J'_{-\nu}(z) \end{vmatrix} = -\frac{2\sin\nu\pi}{\pi z}, \quad z \neq 0.$$

(A.1.6) $$zJ'_\nu(z) \pm \nu J_\nu(z) = \pm zJ_{\nu\mp 1}(z).$$

(A.1.7) $$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{+1} (1-s^2)^{\nu-\frac{1}{2}} e^{izs}\,ds, \quad \mathrm{Re}\,\nu > -\tfrac{1}{2}.$$

This formula can be interpreted for $\nu \neq -\frac{1}{2}, -\frac{3}{2}$ etc. in distributional sense. See Gelfand/Schilow [10].

Finally we mention the Hankel functions, which have some use in section A.3.

For $\nu \notin \mathbb{Z}$:

$$H_\nu^{(1)}(z) := \frac{J_{-\nu}(z) - e^{-\nu\pi i}J_\nu(z)}{i\sin\nu\pi},$$

$$H_\nu^{(2)}(z) := \frac{e^{\nu\pi i}J_\nu(z) - J_{-\nu}(z)}{i\sin\nu\pi}.$$

So $J_\nu(z) = \frac{1}{2}(H_\nu^{(1)}(z) + H_\nu^{(2)}(z))$.

A.2. Airy functions.

In this section we summarize some wellknown facts about Airy functions. See also Erdélyi [8].

The function Ai(z) is defined for z real by

(A.2.1) $$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(zs+\frac{1}{3}s^3)}\,ds.$$

It can be shown that Ai(z) is an entire function in z which solves the Airy equation

(A.2.2) $$\frac{d^2u}{dz^2} - zu = 0.$$

It can be written as

(A.2.3) $\quad Ai(z) = Ai(0)y_1(z) + Ai'(0)y_2(z)$

with $y_1(z) = \Sigma_{m=0}^{\infty} \alpha_m z^{3m}$ and $y_2(z) = z\,\Sigma_{m=0}^{\infty} \beta_m z^{3m}$ entire.

$$Ai(0) = \frac{1}{3^{\frac{2}{3}}\Gamma(\frac{2}{3})}, \quad Ai'(0) = -\frac{1}{3^{\frac{1}{3}}\Gamma(\frac{1}{3})} .$$

The functions $Ai(e^{\pm 2\pi i/3}z)$ are solutions of equation (A.2.2) as well. Then $Ai(z) = e^{-\pi i/3}Ai(e^{2\pi i/3}z) + e^{\pi i/3}Ai(e^{-2\pi i/3}z)$. If we define

(A.2.4) $$Bi(z) = i\left[e^{-\pi i/3}Ai(e^{2\pi i/3}z) - e^{\pi i/3}Ai(e^{-2\pi i/3}z)\right]$$

then $Ai(z)$ and $Bi(z)$ are solutions which are real for real z.

(A.2.5) $$\begin{vmatrix} Ai(z) & Bi(z) \\ Ai'(z) & Bi'(z) \end{vmatrix} = \frac{1}{\pi} .$$

Finally for $x > 0$:

(A.2.6) $$Ai(0)Bi(x) - Bi(0)Ai(x) = 2Ai(0)\left(\frac{x}{3}\right)^{\frac{1}{2}} e^{-\pi i/6}\, J_{\frac{1}{3}}(\tfrac{2}{3}ix^{\frac{3}{2}}).$$

A.3. LEMMA A.3.1. *Let* $\Omega \subset \mathbb{R}^n$, Ω *open. Let* $\psi = \psi(x,\xi) : \Omega \times (\mathbb{R}^N \setminus \{0\}) \to \mathbb{R}^+$ *satisfy* $S^{\lambda}_{1,0}$*-estimates in* $\Omega \times (\mathbb{R}^N \setminus \{0\})$, $\lambda > 0$, *such that for every* K *compact,* $K \subset \Omega$ *there are constants* $c_1, c_2 > 0$ *such that* $\forall\, x \in K$, $\xi \neq 0$: $c_1|\xi|^{\lambda} \le \psi(x,\xi) \le c_2|\xi|^{\lambda}$. *Let* $a = a(y)$ *be a smooth function on* $\mathbb{R}^+$ *such that* $\exists\, m_0$: $\forall\, n$: $\exists\, C > 0$: $\forall\, y \ge 1$: $|a^{(n)}(y)| \le Cy^{m_0-n}$. *Let* $\chi = \chi(s)$ *be a* C^{∞} *function such that* $0 \le \chi \le 1$ *and*

$$\chi = \begin{cases} 1 & |s| \ge M \\ 0 & |s| \le N \end{cases}, \quad 0 < N < M < \infty.$$

Then $b(x,\xi) := \chi(|\xi|)a(\psi(x,\xi))$ *is an element of* $S^{\lambda m_0}_{1,0}(\Omega \times \mathbb{R}^N)$.

PROOF. Clearly b is smooth on $\Omega \times \mathbb{R}^N$. Without restriction we can from now on assume $|\xi| \ge N$ and omit χ. The method of proof will be a familiar one. See Melrose [19].

Let K be a compact set, $K \subset \Omega$. By induction with respect to n we will prove that for all $n \ge 0$

(A.3.2) $$\forall\, j: |\gamma_1| + |\gamma_2| \le n \Rightarrow$$
$$\forall\, x \in K: |D_x^{\gamma_1} D_{\xi}^{\gamma_2} a^{(j)}(\psi(x,\xi))| \le C|\xi|^{\lambda(m_0-j)-|\gamma_2|}.$$

For $j = 0$ we then obtain the desired result.

The basisstep $n = 0$ is shown by remarking that the estimates for ψ and $a^{(j)}$ show that $|a^{(j)}(\psi(x,\xi))| \le C|\xi|^{(m_0-j)\lambda}$ for $|\xi| \ge N$.

Now we assume estimate (A.3.2) to be proved for $n \le m$.
With $D = \partial/\partial x_k$ or $D = \partial/\partial\xi_k$ and $|\gamma_1| + |\gamma_2| \le m$ then

$$\text{(A.3.3)} \qquad DD_x^{\gamma_1}D_\xi^{\gamma_2}a^{(j)}(\psi(x,\xi)) = D_x^{\gamma_1}D_\xi^{\gamma_2}\Big[a^{(j+1)}(\psi(x,\xi))\cdot D\psi(x,\xi)\Big].$$

Because $a^{(j+1)}(\psi(x,\xi))$ behaves like a symbol in $S_{1,0}^{\lambda(m_0-j-1)}$ at least for $|\gamma_1| + |\gamma_2| \le m$, and $D\psi$ satisfies $S_{1,0}^{\lambda}$- or $S_{1,0}^{\lambda-1}$-estimates, the multiplicative property of symbols shows that (A.3.3) satisfies estimate (A.3.2) for $n = m+1$. □

LEMMA A.3.4. *Let* a,ψ,χ *and* Ω *be as in* Lemma A.3.1. *Assume moreover that*

$$\exists\ y_0 < \infty,\ c_0 > 0:\ \forall\ y \ge y_0:\ |y^{-m_0}a(y)| \ge c_0.$$

Then $b(x,\xi) = \chi(|\xi|)a(\psi(x,\xi))$ *is an elliptic element of* $S_{1,0}^{\lambda m_0}(\Omega \times \mathbb{R}^N)$ *of order* λm_0.

PROOF. Only ellipticity has to be shown. Since $|a(y)| \ge c_0|y|^{m_0}$ for $y \ge y_0$ we have for $x \in K$ compact, $|\xi| \ge \xi_0$, $0 < \xi_0 < \infty$ and some $c > 0$:

$$|a(\psi(x,\xi))| \ge c|\xi|^{\lambda m_0}.$$

This shows that $b(x,\xi)$ is elliptic of order λm_0. □

EXAMPLE. Let a be smooth for $y > 0$ so that

$$a(y) \sim \sum_{n=0}^{\infty} a_n y^{m-n}, \quad a_0 \ne 0, \quad y \to \infty,$$

and an asymptotic expansion for $a^{(n)}$ is obtained by differentiation of the series. Then a satisfies the assumptions of Lemma A.3.4.

A.4. Asymptotic expansions of Bessel functions and related symbols.

The asymptotic expansion for $J_\nu(z)$ can be derived from the formula

$$J_\nu(z) = \tfrac{1}{2}\Big[H_\nu^{(1)}(z) + H_\nu^{(2)}(z)\Big]$$

and the asymptotic expansions for $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$. In Watson [27] we find

$$H_\nu^{(1)}(z) \sim e^{iz}\sum_{n=0}^{\infty} h_{\nu,n}^{(1)} z^{-\frac{1}{2}-n}, \quad h_{\nu,n}^{(1)} = \sqrt{\frac{2}{\pi}}\,e^{-i(\frac{1}{2}\nu+\frac{1}{4})\pi}\,\frac{(-1)^n(\nu,n)}{(2i)^n},$$

uniformly in z for $|z| \to \infty$, $-\pi < \arg z < 2\pi$ and

$$H_\nu^{(2)}(z) \sim e^{-iz} \sum_{n=0}^{\infty} h_{\nu,n}^{(2)} z^{-\frac{1}{2}-n}, \quad h_{\nu,n}^{(2)} = \sqrt{\frac{2}{\pi}}\, e^{i(\frac{1}{2}\nu+\frac{1}{4})\pi} \frac{(\nu,n)}{(2i)^n},$$

uniformly in z for $|z| \to \infty$, $-2\pi < \arg z < \pi$. Asymptotic expansions for derivatives of $e^{-iz}H_\nu^{(1)}(z)$ and $e^{iz}H_\nu^{(2)}(z)$ can be obtained by formal differentiation of the asymptotic series. Expansions valid in other sectors are provided by the relations:

$$H_\nu^{(1)}(ze^{m\pi i}) = \frac{\sin(1-m)\nu\pi}{\sin \nu\pi} H_\nu^{(1)}(z) - e^{-\nu\pi i} \frac{\sin m\nu\pi}{\sin \nu\pi} H_\nu^{(2)}(z),$$

$$H_\nu^{(2)}(ze^{m\pi i}) = \frac{\sin(1+m)\nu\pi}{\sin \nu\pi} H_\nu^{(2)}(z) + e^{\nu\pi i} \frac{\sin m\nu\pi}{\sin \nu\pi} H_\nu^{(1)}(z).$$

Here $\arg(ze^{m\pi i}) = m\pi + \arg z$, $-\pi < \arg z \le \pi$. In particular, an expansion for $H_\nu^{(2)}(z)$ valid in the sector $0 < \arg z < 2\pi$ is given by

$$H_\nu^{(2)}(z) \sim e^{-iz} \sum_{n=0}^{\infty} h_{\nu,n}^{(2)} z^{-\frac{1}{2}-n} - e^{\nu\pi i}\, 2\cos\nu\pi\, e^{iz} \sum_{n=0}^{\infty} h_{\nu,n}^{(1)} z^{-\frac{1}{2}-n}.$$

Finally we remark that

$$H_\nu^{(2)}(z) + e^{\nu\pi i}\, 2\cos\nu\pi\, H_\nu^{(1)}(z) \sim e^{-iz} \sum_{n=0}^{\infty} h_{\nu,n}^{(2)} z^{-\frac{1}{2}-n}$$

for $|z| \to \infty$, $0 < \arg z < 2\pi$.

In chapter 5 we are interested in the asymptotic behaviour of $J_\nu(2i|\xi|t^{\frac{1}{2}})$ for $|\xi| \to \infty$, $t \in \mathbb{R}\setminus\{0\}$, $\arg t = 0$ for $t > 0$ and $\arg t = \pi$ for $t < 0$. We define for $z \neq 0$, $0 < \arg z < 2\pi$

$$a_\nu^+(z) := e^{-iz}H_\nu^{(1)}(z),$$

$$a_\nu^-(z) := e^{iz}\Big[H_\nu^{(2)}(z) + e^{\nu\pi i}\, 2\cos\nu\pi\, H_\nu^{(1)}(z)\Big].$$

For $t \neq 0$ and $\xi \neq 0$ then

$$h_\nu^+(t,\xi) := a_\nu^+(2i|\xi|t^{\frac{1}{2}}) \text{ and } h_\nu^-(t,\xi) := a_\nu^-(2i|\xi|t^{\frac{1}{2}}).$$

Now

$$a_\nu^+(z) \sim \sum_{n=0}^{\infty} h_{\nu,n}^{(1)} z^{-\frac{1}{2}-n} \quad \text{and} \quad a_\nu^-(z) \sim \sum_{n=0}^{\infty} h_{\nu,n}^{(2)} z^{-\frac{1}{2}-n}.$$

Let $\psi(t,\xi) := 2|t|^{\frac{1}{2}}|\xi|$ for $t\xi \neq 0$. Then $a^{\pm}$ and ψ satisfy the conditions of Lemma A.3.1, so with χ an arbitrary cut-off function:

$$\chi(|\xi|)h_\nu^+(t,\xi) \text{ and } \chi(|\xi|)h_\nu^-(t,\xi)$$

are elements of $S_{1,0}^{-\frac{1}{2}}((\mathbb{R}\setminus\{0\}) \times \mathbb{R}^n)$.

Moreover, Lemma (A.3.4) and the fact that $h^{(1)}_{\nu,0}$ and $h^{(2)}_{\nu,0}$ are unequal to zero show that these symbols are elliptic of order $-\frac{1}{2}$.

The next two lemmas give us estimates which will be useful in chapter 5 as well.

LEMMA A.4.1. *Let* $\chi : \mathbb{C} \to \mathbb{R}$ *be smooth,* $0 \le \chi \le 1$ *and*

$$\chi(z) = \begin{cases} 1 & \text{for } |z| \ge M \\ 0 & \text{for } |z| \le N \end{cases}, \quad 0 < N < M < \infty$$

and define

$$\ell^{\pm}_{\nu}(t,\xi) := \chi(2i|\xi|t^{\frac{1}{2}})h^{\pm}_{\nu}(t,\xi).$$

Then ℓ^{+}_{ν} *and* ℓ^{-}_{ν} *are smooth in* (t,ξ). *Let* K *be a compact subset of* $\mathbb{R}$. *Then*

(A.4.2) $$\forall\ k:\ \forall\ \gamma:\ \exists\ C:\ |D^k_t D^{\gamma}_{\xi}\ell^{\pm}_{\nu}(t,\xi)| \le C(1+|\xi|)^{2k} \qquad \textit{for } t \in K,\ \xi \in \mathbb{R}^n.$$

PROOF. Of course ℓ^{+}_{ν} and ℓ^{-}_{ν} are smooth. The rest of the proof is quite similar to the proof of Lemma (A.3.1). So we will show by induction with respect to n that for all $n \ge 0$

(A.4.3) $$\forall\ k+|\gamma| \le n:\ \forall\ j:\ \exists\ C:\ |D^k_t D^{\gamma}_{\xi}(\chi a^{\pm}_{\nu})^{(j)}(2i|\xi|t^{\frac{1}{2}})| \le \le C(1+|\xi|)^{2k} \qquad \text{for } t \in K,\ \xi \in \mathbb{R}^n.$$

For $n = 0$ it is sufficient to note that $(\chi a^{\pm}_{\nu})^{(j)}(2i|\xi|t^{\frac{1}{2}})$ is bounded for $2|t|^{\frac{1}{2}}|\xi| \ge N$. So suppose estimate (A.4.3) is true for $n \le m$. Then for $k + |\gamma| \le m$

(A.4.4) $$DD^k_t D^{\gamma}_{\xi}(\chi a^{\pm}_{\nu})^{(j)}(2i|\xi|t^{\frac{1}{2}}) = D^k_t D^{\gamma}_{\xi}\Big[(\chi a^{\pm}_{\nu})^{(j+1)}(2i|\xi|t^{\frac{1}{2}})\cdot D(2i|\xi|t^{\frac{1}{2}})\Big].$$

It is easy to show that $2i|\xi|t^{\frac{1}{2}}$ satisfies estimates such as in expression (A.4.2) for $2|\xi||t|^{\frac{1}{2}} \ge N$, $k + |\gamma| \ge 1$.
An application of Leibniz' rule then gives estimate (A.4.3) with $n = m+1$. □

REMARK. The estimates (A.4.2) do not show that $\ell^{\pm}$ is a symbol. For $\gamma = 0$ it is easily shown that these estimates cannot be improved. So $\ell^{\pm}$ is not a symbol at all. This is different from the case of the Airy function, in which a similar construction did give a symbol (see Melrose [19]).

LEMMA A.4.5. $\forall\ k \ge 0:\ \exists\ C_k:\ \forall\ (\xi,t):\ |t|^{\frac{1}{2}}|\xi| \ge 1 \Rightarrow$

$$\left|e^{\pm 2|\xi|t^{\frac{1}{2}}}\frac{\partial^k}{\partial t^k}e^{\mp 2|\xi|t^{\frac{1}{2}}}\right| \le C_k|\xi|^{2k}.$$

PROOF. $$\frac{\partial^{m+1}}{\partial t^{m+1}} e^{\mp 2|\xi|t^{\frac{1}{2}}} = \frac{\partial^m}{\partial t^m}\left[\mp|\xi|t^{-\frac{1}{2}}e^{\mp 2|\xi|t^{\frac{1}{2}}}\right].$$

Because
$$\left|\frac{\partial^k}{\partial t^k}(|\xi|t^{-\frac{1}{2}})\right| \le C|\xi|^{2k+2} \quad \text{for } |t|^{\frac{1}{2}}|\xi| \ge 1,$$

an induction argument gives the desired result. □

COROLLARY A.4.6. *Let* K *be a compact subset of* $\mathbb{R}$.
$$e^{\pm 2|\xi|t^{\frac{1}{2}}}\ell_\nu^\pm(\xi,t)$$
is smooth and
$$\forall\, k \ge 0 : \exists\, C_k : \left|\frac{\partial^k}{\partial t^k} e^{\pm 2|\xi|t^{\frac{1}{2}}}\ell_\nu^\pm(\xi,t)\right| \le C_k(1+|\xi|)^{2k}\left|e^{\pm 2|\xi|t^{\frac{1}{2}}}\right|$$
for $t \in K$ *and every* ξ.

PROOF. This follows from Lemma (A.4.1), Lemma (A.4.5) and Leibniz' rule. □

A.5. Estimates for the Airy function and related symbols.

In this section we give estimates for the functions $Ai(\omega t|\xi|^{\frac{2}{3}})$, $\omega \in \mathbb{C}$ fixed, $|\omega| = 1$, which appear in the discussion of the Tricomi operator. We remark here that in Taylor [24] and Melrose [19] estimates are given for expressions involving Airy functions which are similar to the ones we give. However, in order to have them at our disposal when we need them, we state and prove the estimates for exactly the expressions we encounter.

An asymptotic expansion for $Ai(z)$ valid for $z \in \mathbb{C}$, $|\arg z| < \pi$ has the form
$$Ai(z) = e^{-\frac{2}{3}z^{\frac{3}{2}}} ai(z) \quad \text{with } ai(z) \sim \sum_{n=0}^{\infty} a_n z^{-\frac{1}{4}-3n/2}, \; a_0 \neq 0.$$

Asymptotic expansions for derivatives of $ai(z)$ can be obtained by differentiation of the series. An expansion for $Ai(z)$ valid in a sector containing z with $\arg z = \pi$ is found by using the relation
$Ai(z) = e^{\pi i/3}Ai(ze^{-2\pi i/3}) + e^{-\pi i/3}Ai(ze^{2\pi i/3})$.

Let $\omega \in \mathbb{C}$, $|\omega| = 1$, $|\arg \omega| < \pi$ and define
$$ai_\omega(t,\xi) := ai(\omega|t||\xi|^{\frac{2}{3}}), \quad \xi \in \mathbb{R}^N.$$

LEMMA A.5.1. *Let* χ *be as in* Lemma (A.3.1). *Then* $\chi(|\xi|)ai_\omega(t,\xi)$ *is an elliptic element of* $S^{-\frac{1}{6}}_{1,0}((\mathbb{R}\setminus\{0\}) \times \mathbb{R}^N)$.

PROOF. $\psi(t,\xi) = |t||\xi|^{\frac{2}{3}}$ satisfies the conditions of Lemma (A.3.1) with $\lambda = \frac{2}{3}$. $a(y) = ai(\omega y)$ satisfies the conditions of Lemma (A.3.4) with $m = -\frac{1}{4}$. This follows from the asymptotic expansion of ai(z). □

In particular we define for $t \neq 0$:

$$a_+(t,\xi) := ai(e^{2\pi i/3}t|\xi|^{\frac{2}{3}}),$$
$$a_-(t,\xi) := ai(e^{-2\pi i/3}t|\xi|^{\frac{2}{3}}).$$

LEMMA A.5.2. 1) *Let* $\xi \in \mathbb{R}^N$, $\omega \in \mathbb{C}$, $|\omega| = 1$, $|\arg \omega| < \pi$, $T > 0$. *Then*

$$\forall\, n \geq 0: \exists\, C_n: \left|\frac{\partial^n}{\partial t^n} Ai(\omega t|\xi|^{\frac{2}{3}})\right| \leq C_n(1+|\xi|)^{2n}\left|e^{-\frac{2}{3}|\xi|(\omega t)^{\frac{3}{2}}}\right|$$
for $0 \leq t \leq T$.

2) *a similar statement holds for* $-T \leq t \leq 0$ *and* ω *such that* $|\arg(\omega t)| < \pi$.

PROOF. $\frac{\partial^n}{\partial t^n} Ai(\omega t|\xi|^{\frac{2}{3}}) = (\omega|\xi|^{\frac{2}{3}})^n Ai^{(n)}(\omega t|\xi|^{\frac{2}{3}})$ so we must show that

$$|Ai^{(n)}(\omega t|\xi|^{\frac{2}{3}})| \leq C_n(1+|\xi|)^{n/3}\left|e^{-\frac{2}{3}|\xi|(\omega t)^{\frac{3}{2}}}\right|.$$

Now $Ai^{(2)}(z) = zAi(z)$ so for $n \geq 3$: $Ai^{(n)}(z) = zAi^{(n-2)}(z) + (n-2)Ai^{(n-3)}(z)$. Therefore it is sufficient to prove this for $n = 0$ and $n = 1$. For $n = 0$ it is obvious because $ai(\omega y)$ is continuous and bounded for $y \to \infty$.

$$Ai^{(1)}(z) = e^{-\frac{2}{3}z^{\frac{3}{2}}}\left[-z^{\frac{1}{2}}ai(z) + ai^{(1)}(z)\right].$$

Here $ai^{(1)}(\omega y)$ is continuous and $|-(\omega y)^{\frac{1}{2}}ai(\omega y) + ai^{(1)}(\omega y)| \leq Cy^{\frac{1}{4}}$ for $y \geq 1$. Hence the result. □

For the function Bi(z) we remark that for $|\arg z| < \frac{\pi}{3}$

$$Bi(z) = e^{\frac{2}{3}z^{\frac{3}{2}}} bi(z)$$

in which bi(z) has an asymptotic expansion of the form

$$bi(z) \sim \sum_{n=0}^{\infty} b_n z^{-\frac{1}{4}-3n/2}, \quad b_0 \neq 0.$$

For $bi(t|\xi|^{\frac{2}{3}})$ similar conclusions hold as given in Lemma (A.5.1).

A.6. Convolution with the distribution $vp\frac{1}{x}$.

In this section we derive some properties for the convolution of functions with the distribution $vp\frac{1}{x}$ that will be used in section 4.5.

For $\varphi \in C_0^\infty(\mathbb{R})$

$$\langle vp\frac{1}{x}, \varphi\rangle = \lim_{\varepsilon\downarrow 0} \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x}dx = \lim_{\varepsilon\downarrow 0} \int_{-M}^{-\varepsilon} + \int_{\varepsilon}^{M} \frac{\varphi(x)}{x}dx$$

if supp $\varphi \subset [-M,+M]$. Because

$$0 = \int_{-M}^{-\varepsilon} + \int_{\varepsilon}^{M} \frac{\varphi(0)}{x}dx$$

and $\frac{\varphi(x)-\varphi(0)}{x}$ is continuous this is welldefined. This also holds for $\varphi \in C_0^1(\mathbb{R})$.

For $\varphi \in C^1(\mathbb{R}) \cap L_2(\mathbb{R})$ we can define

$$\langle vp\frac{1}{x}, \varphi\rangle := \int_{-\infty}^{-1} + \int_{+1}^{\infty} \frac{\varphi(x)}{x}dx + \lim_{\varepsilon\downarrow 0} \int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1} \frac{\varphi(x)}{x}.$$

If $\varphi_j \to 0$ in $C^1(\mathbb{R})$ and in $L_2(\mathbb{R})$, $\langle vp\frac{1}{x}, \varphi_j\rangle \to 0$.

For φ as above we will discuss the convolution $vp\frac{1}{x} * \varphi = \langle vp\frac{1}{y}, \varphi(x-y)\rangle$.

LEMMA A.6.1. *Let* $\varphi \in C^1(\mathbb{R}) \cap L_2(\mathbb{R})$ *and* $\frac{d\varphi}{dx}$ *bounded. Then* $vp\frac{1}{x} * \varphi$ *is a bounded and continuous function.*

PROOF.

$$\left|\int_{|y|\geq 1} \frac{\varphi(x-y)}{y}dy\right| \leq \int_{|y|\geq 1} \left|\frac{\varphi(x-y)}{y}\right|dy \leq$$

$$\leq \left(\int_{|y|\geq 1} \frac{1}{y^2}dy \int |\varphi(x-y)|^2 dy\right)^{\frac{1}{2}} \leq M < \infty, \text{ M independent of x.}$$

$$\left|\int_{\varepsilon\leq|y|\leq 1} \frac{\varphi(x-y)}{y}dy\right| = \left|\int_{\varepsilon\leq|y|\leq 1} \frac{\varphi(x-y)-\varphi(x)}{y}dy\right| \leq$$

$$\leq 2 \max_{|\xi-x|\leq 1} |\varphi'(\xi)| \leq 2M_0 < \infty.$$

□

LEMMA A.6.2. *Let* $\varphi \in C_0^1(\mathbb{R})$. *Then* $vp\frac{1}{\xi} * \varphi(\xi) = u_0 + s$ with $u_0 \in C_0^0$ and $s \in S_{1,0}^{-1}$.

PROOF. Let $\psi_1 \in C_0^\infty(\mathbb{R})$ be an even function so that

$$\psi_1(\xi) = \begin{cases} 1 & \text{for } |\xi| < \frac{1}{2} \\ 0 & \phantom{\text{for }} |\xi| > 1 \end{cases}$$

and $\psi_2 := 1-\psi_1$. Then $u_0 := \psi_1\, vp\frac{1}{\xi} * \varphi$ and $s := \psi_2\, vp\frac{1}{\xi} * \varphi$. u_0 is in C_0^0, s is clearly smooth and

$$D^n \psi_2\, vp\frac{1}{\xi} * \varphi = \int D_\xi^n\left(\frac{\psi_2(\xi-\eta)}{\xi-\eta}\right)\varphi(\eta)d\eta.$$

Here $|\eta| \leq R$ so that for $|\xi| \geq R+1$: $|\xi-\eta| \geq ||\xi|-|\eta|| = |\xi|-|\eta| \geq |\xi|-R \geq 1$.

Then

$$\left|D^n \frac{\psi_2(\xi-\eta)}{\xi-\eta}\right| = \left|D^n \frac{1}{\xi-\eta}\right| = C\left|\frac{1}{(\xi-\eta)^{n+1}}\right| \le C(|\xi| - R)^{-(n+1)}. \qquad \square$$

LEMMA A.6.3. *Let* $\varphi = \varphi(\xi) \in S^m_{\rho,0}(\mathbb{R})$ *with* $m < -\frac{1}{2}$. *Then* $\mathrm{vp}\frac{1}{\xi} * \varphi \in S^{m_0}_{\frac{1}{2}\rho,0}$ *with* $m_0 > \frac{1}{2} + \frac{\rho}{2} + \max(m,-1)$.

PROOF. Let ψ_1 and ψ_2 be as in the previous Lemma. Then $\psi_1\,\mathrm{vp}\frac{1}{\xi} * \varphi$ is smooth. So is $\psi_2\,\mathrm{vp}\frac{1}{\xi} * \varphi$ because of Lemma A.6.6.

$$D^n\psi_1\,\mathrm{vp}\frac{1}{\xi} * \varphi = \mathrm{vp}\int \psi_1(\eta)\frac{(D^n\varphi)(\xi-\eta)}{\eta}\,d\eta =$$

$$= \int \psi_1(\eta)\,\frac{(D^n\varphi)(\xi-\eta) - (D^n\varphi)(\xi)}{\eta}\,d\eta$$

and

$$\frac{(D^n\varphi)(\xi-\eta) - (D^n\varphi)(\xi)}{-\eta} = (D^{n+1}\varphi)(\xi-\theta) \text{ with } \theta \in (-1,+1)$$

But then $|D^{n+1}\varphi(\xi-\theta)| \le C(|\xi|-1)^{m-\rho(n+1)}$ for $|\xi| \ge 2$.
Further, $\psi_2\,\mathrm{vp}\frac{1}{\xi} \in S^{-1}_{1,0}$ and $\varphi \in S^m_{\rho,0}$ so that $\psi_2\,\mathrm{vp}\frac{1}{\xi} * \varphi \in S^{\frac{1}{2}+\rho/2+\max(-1,m)+\varepsilon}_{\frac{1}{2}\rho,0}$ for every $\varepsilon > 0$ as follows from Lemma A.6.7.
Therefore $\mathrm{vp}\frac{1}{\xi} * \varphi$ is in $S^{\frac{1}{2}+\rho/2+\max(-1,m)+\varepsilon}_{\frac{1}{2}\rho,0}$ for every $\varepsilon > 0$. $\square$

It is wellknown that in $\mathcal{D}'(\mathbb{R})$ (and even in S')

$$\frac{\varepsilon}{\pi(x^2+\varepsilon^2)} \to \delta_{(x=0)} \text{ and } \frac{1}{x \pm i\varepsilon} \to \mathrm{vp}\frac{1}{x} \mp i\pi\delta_{(x=0)} \text{ for } \varepsilon \downarrow 0.$$

LEMMA A.6.4. *Let* $\varphi \in L_2(\mathbb{R})$. *Then* $\frac{1}{x \pm i\varepsilon} * \varphi$ *is an analytic function of* $z = x \pm i\varepsilon$ *for* $\mathrm{Im}\, z \gtrless 0$. *It is bounded for* $\mathrm{Im}\, z \gtrless \pm\delta$, $\delta > 0$ *arbitrary.*

PROOF. This Lemma is wellknown. See for instance Hochstadt [14], page 191, Theorem 7. $\square$

LEMMA A.6.5. *Let* $\varphi \in C^1(\mathbb{R}) \cap L_2(\mathbb{R})$ *and* $\frac{d\varphi}{dx}$ *bounded. Then* $\frac{1}{x \pm i\varepsilon} * \varphi \to \mathrm{vp}\frac{1}{x} * \varphi \mp i\pi\varphi$ *for* $\varepsilon \downarrow 0$ *in the supremum norm.*

PROOF. We only discuss the convergence of $\frac{1}{x+i\varepsilon} * \varphi$. The other sign goes similarly. From Lemma A.6.1 we know that $\mathrm{vp}\frac{1}{x} * \varphi$ is a bounded function and Lemma A.6.4 shows that $\frac{1}{x+i\varepsilon} * \varphi$ is a bounded function for $\varepsilon > 0$ fixed.

$$\frac{1}{x+i\varepsilon} * \varphi - \mathrm{vp}\frac{1}{x} * \varphi + i\pi\varphi =$$

$$\frac{1}{x+i\varepsilon} * \varphi - vp\frac{1}{x} * \varphi + \frac{i\varepsilon}{x^2+\varepsilon^2} * \varphi - \frac{i\varepsilon}{x^2+\varepsilon^2} * \varphi + i\pi\varphi .$$

Now

$$\frac{i\varepsilon}{x^2+\varepsilon^2} = \frac{1}{2}\left(\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon}\right) \text{ and } \frac{1}{2}\left(\frac{1}{x-i\varepsilon} + \frac{1}{x+i\varepsilon}\right) = \frac{x}{x^2+\varepsilon^2}$$

so

$$\frac{1}{x+i\varepsilon} * \varphi - vp\frac{1}{x} * \varphi + \frac{i\varepsilon}{x^2+\varepsilon^2} * \varphi = \frac{x}{x^2+\varepsilon^2} * \varphi - vp\frac{1}{x} * \varphi$$

and

$$\left|\int_{|y|\geq 1} \left(\frac{y}{y^2+\varepsilon^2} - \frac{1}{y}\right)\varphi(x-y)dy\right| = \left|\int_{|y|\geq 1} \frac{-\varepsilon^2}{y(y^2+\varepsilon^2)}\varphi(x-y)dy\right| \leq$$

$$\leq \varepsilon^2\left(\int_{|y|\geq 1} \frac{dy}{y^6} \int |\varphi(x-y)|^2 dy\right)^{\frac{1}{2}} \leq C\varepsilon^2,$$

$$\left|\int_{\delta\leq|y|\leq 1} \frac{-\varepsilon^2}{y(y^2+\varepsilon^2)}\varphi(x-y)dy\right| =$$

$$\left|\int_{\delta\leq|y|\leq 1} \frac{-\varepsilon^2}{y(y^2+\varepsilon^2)}(\varphi(x-y)-\varphi(x))dy\right| \leq M\varepsilon^2 \int_{|y|\leq 1} \frac{dy}{y^2+\varepsilon^2} \leq \varepsilon M\pi$$

because $|\frac{1}{y}(\varphi(x-y)-\varphi(x))| = |\varphi'(\xi)| \leq M$.

$$\frac{i\varepsilon}{x^2+\varepsilon^2} * \varphi - i\pi\varphi = i\varepsilon \int \frac{\varphi(x-y)-\varphi(x)}{y^2+\varepsilon^2}dy$$

and

$$\left|\varepsilon \int_{|y|\leq\varepsilon} \frac{\varphi(x-y)-\varphi(x)}{y^2+\varepsilon^2}dy\right| \leq \varepsilon M \int_{|y|\leq\varepsilon} \frac{|y|dy}{y^2+\varepsilon^2} \leq \varepsilon M\pi,$$

$$\left|\varepsilon \int_{\varepsilon\leq|y|\leq 1} \frac{\varphi(x-y)-\varphi(x)}{y^2+\varepsilon^2}dy\right| \leq \varepsilon \int_{\varepsilon\leq|y|\leq 1} \frac{|y|M}{y^2}dy = -2\varepsilon M\log\varepsilon,$$

$$\left|\varepsilon \int_{|y|\geq 1} \frac{\varphi(x-y)-\varphi(x)}{y^2+\varepsilon^2}dy\right| \leq \varepsilon \int_{|y|\geq 1} \frac{2M_0}{y^2}dy = 4\varepsilon M_0,$$

$M_0 = \max |\varphi(y)|$. □

<u>LEMMA A.6.6</u>. *Let* f *and* g *belong to* $L_2(\mathbb{R})$.

1. $f * g$ *is continuous.*
2. *If* $f \in C^1(\mathbb{R})$ *and for some* $\alpha > \frac{1}{2}$: $|f'(x)| \leq C|x|^{-\alpha}$ *for* $|x| \geq 1$, *then* $f * g \in C^1(\mathbb{R})$ *and* $\frac{d}{dx} f * g = f' * g$.

<u>PROOF</u>. 1. $|(f * g)(x+h) - (f * g)(x)| \leq \int |f(x+h-y) - f(x-y)|\,|g(y)|\,dy$

$= \int |f(y+h) - f(y)|\,|g(x-y)|dy \leq \|f_h - f\|_2\|g\|_2 \to 0$ if $h \to 0$.

Here $f_h(y) := f(y+h)$. (See Rudin [21], page 196, Theorem 9.5).

2. Fix $\varepsilon \neq 0$, $|\varepsilon| \le 1$. Then

$$\left|\frac{f(y+\varepsilon) - f(y)}{\varepsilon} - f'(y)\right| = |f'(y+\eta) - f'(y)|$$

for some η, $0 < |\eta| < |\varepsilon|$.
f' is continuous and for $|y| \ge 2$: $|y+\eta| \ge ||y| - |\eta|| = |y| - |\eta| > |y| - 1 \ge 1$.
So $|f'(y+\eta)| \le C(|y|-1)^{-\alpha}$ for $|y| \ge 2$, $|\eta| \le 1$. Therefore
$|f'(y+\eta)| \le h(y)$ with $h \in L_2(\mathbb{R})$ for all $|\eta| \le 1$. Now

$$\left|\frac{(f * g)(x+\varepsilon) - (f * g)(x)}{\varepsilon} - (f' * g)(x)\right| \le$$

$$\le \int \left|\frac{f(y+\varepsilon) - f(y)}{\varepsilon} - f'(y)\right| |g(x-y)| dy$$

and

$$\left|\frac{f(y+\varepsilon) - f(y)}{\varepsilon} - f'(y)\right| |g(x-y)| \le (|h(y)| + |f'(y)|)|g(x-y)|,$$

which is an integrable function. Here h is independent of ε, $0 < |\varepsilon| \le 1$.
An application of Lebesgue's Dominated Convergence Theorem gives the desired result. □

LEMMA A.6.7. *Let* $\sigma = \sigma(\xi)$ *be an element of* $S^{m_1}_{\rho_1,0}(\mathbb{R})$ *and* $\tau = \tau(\xi)$ *an element of* $S^{m_2}_{\rho_2,0}(\mathbb{R})$. *Assume* $0 < \rho_1 \le \rho_2 \le 1$, $m_1 < -\frac{1}{2}$ *and* $m_2 < -\frac{1}{2}$.
Then $\sigma * \tau$ *is an element of* $S^{m}_{\frac{1}{2}\rho_1,0}$ *for* $m > \frac{1}{2} + \rho_1/2 + \max(m_1,m_2)$.

PROOF. $(\sigma * \tau)(\xi) = \int \sigma(\xi-\eta)\tau(\eta)d\eta$ is welldefined and smooth as follows from Lemma A.6.6. For $0 \le k \le n$:

$$D^n_\xi(\sigma * \tau)(\xi) = \int (D^n\sigma)(\xi-\eta)\tau(\eta)d\eta = \int (D^{n-k}\sigma)(\xi-\eta)(D^k\tau)(\eta)d\eta$$

by means of partial integration.
Further for $\alpha \ge 0$: $|\xi|^\alpha \le (|\xi-\eta| + |\eta|)^\alpha \le c_\alpha(|\xi-\eta|^\alpha + |\eta|^\alpha)$ so

$$\left||\xi|^\alpha D^n_\xi(\sigma * \tau)(\xi)\right| \le$$

$$\le c_\alpha \int |\xi-\eta|^\alpha \left|(D^{n-k}\sigma)(\xi-\eta)(D^k\tau)(\eta)\right| d\eta + c_\alpha \int \left|(D^{n-k}\sigma)(\xi-\eta)|\eta|^\alpha(D^k\tau)(\eta)\right| d\eta$$

which is bounded in ξ provided $m_1 - \rho_1(n-k) + \alpha < -\frac{1}{2}$ and $m_2 - \rho_2 k + \alpha < -\frac{1}{2}$.
Here we use the fact that the convolution of two functions which are in L_2 is a bounded function. For $n = 0$ this implies $0 \le \alpha$, $m_1 + \alpha < -\frac{1}{2}$ and $m_2 + \alpha < -\frac{1}{2}$. So $0 \le \alpha < -\frac{1}{2} - \max\{m_1,m_2\}$. For $n > 0$ even we choose $k = \frac{n}{2}$ and

then we can take

$$0 \le \alpha < -\tfrac{1}{2} - \max\{m_1, m_2\} + n\frac{\rho_1}{2}.$$

For n odd we choose $k = \frac{n+1}{2}$. Then $0 \le \alpha$, $\alpha < -\frac{1}{2} - m_1 + n(\rho_1/2) - \rho_1/2$ and $\alpha < -\frac{1}{2} - m_2 + n(\rho_2/2) + \rho_2/2$, so we can take

$$0 \le \alpha < -\tfrac{1}{2} - \max\{m_1, m_2\} - \rho_1/2 + n(\rho_1/2).$$

Summing up, we can take

$$\alpha = -\tfrac{1}{2} - \max\{m_1, m_2\} - \rho_1/2 + n(\rho_1/2) - \varepsilon$$

for some $\varepsilon > 0$.
Note that for $\alpha < 0$: $|\xi|^{\alpha} D^n(\sigma * \tau)$ is clearly bounded for $|\xi| \to \infty$. □

A.7. Some lemmas.

LEMMA A.7.1. *Let* $\Omega \subset \mathbb{R}^{n+1}$, Ω *open. Let* P *be a* ΨDO *on* Ω *with real homogeneous principal symbol* p *which is independent of* x. *Here* (x,t), $x \in \mathbb{R}^n$, *denotes a point in* Ω. *Suppose* $(x_0,t_0,\xi_0,\tau_0) \in \Omega \times (\mathbb{R}^{n+1} \setminus 0)$ *is such that* $p(x_0,t_0,\xi_0,\tau_0) = 0$ *and* $\frac{\partial p}{\partial \tau}(x_0,t_0,\xi_0,\tau_0) \ne 0$. *Then in a neighbourhood of* (x_0,t_0,ξ_0,τ_0) τ *is determined as a* C^∞*-function in* (t,ξ) *such that* $\tau_0 = \tau(t_0,\xi_0)$, $p(x,t,\xi,\tau(t,\xi)) = 0$, τ *is homogeneous of order one in* ξ.

PROOF. Implicit Function Theorem. □

LEMMA A.7.2. *Consider again the situation as described in* Lemma A.7.2. *In a neighbourhood of* $((x_0,t_0,\xi_0,\tau_0),(x_0,t_0,\xi_0,\tau_0))$ *the bicharacteristic relation of* P *is given by* Λ'_{φ_0}, *where* φ_0 *is the phase function given by*

$$\varphi_0(x,y,t,s,\theta) = \langle x-y,\theta\rangle + \psi(t,\theta) - \psi_0(s,\theta).$$

Here ψ *is a smooth function near* (t_0,ξ_0), *homogeneous of order one, so that* $\frac{\partial\psi}{\partial t} = \tau(t,\xi)$, $\psi_0(s,\xi) = \psi(s,\xi)$.

PROOF. $\Lambda'_{\varphi_0} = \left\{\left(x,t,\theta,\frac{\partial\psi}{\partial t}; y,s,\theta,\frac{\partial\psi_0}{\partial s}\right) \mid x - y + \frac{\partial\psi}{\partial\theta} - \frac{\partial\psi_0}{\partial\theta} = 0\right\}$.

The Hamilton-Jacobi equations are (with parameter z along a strip):

$$\frac{dx}{dz} = \frac{\partial p}{\partial\xi},\quad \frac{dt}{dz} = \frac{\partial p}{\partial\tau},\quad \frac{d\xi}{dz} = 0,\quad \frac{d\tau}{dz} = -\frac{\partial p}{\partial t},\quad p(x,t,\xi,\tau) = 0.$$

Note that $\frac{\partial\psi}{\partial t} = \tau(t,\theta)$, $\frac{\partial\psi_0}{\partial s} = \tau(s,\theta)$.

It is sufficient to check that $(y - \frac{\partial\psi}{\partial\theta}(t,\theta) + \frac{\partial\psi_0}{\partial\theta}(s,\theta), t, \theta, \tau(t,\theta))$ is on the same integral curve as $(y,s,\theta,\tau(s,\theta))$. For $t = s$ the two points coincide. Since $\frac{\partial p}{\partial\tau} \neq 0$ we can use t as parameter along the strip instead of z.

Then

$$\frac{d}{dt}\left[y - \frac{\partial\psi}{\partial\theta}(t,\theta) + \frac{\partial\psi_0}{\partial\theta}(s,\theta)\right] = -\frac{\partial^2\psi}{\partial t\partial\theta} = -\frac{\partial\tau}{\partial\theta} = \frac{\partial p}{\partial\theta}\Big/\frac{\partial p}{\partial\tau},$$

$$\frac{d}{dt}\left[t\right] = 1 = \frac{\partial p}{\partial\tau}\Big/\frac{\partial p}{\partial\tau},$$

$$\frac{d}{dt}\left[\tau(t,\xi)\right] = \frac{\partial\tau}{\partial t} = -\frac{\partial p}{\partial t}\Big/\frac{\partial p}{\partial\tau}. \qquad \square$$

<u>LEMMA A.7.3</u>. *Let* $u_1,\dots,u_n \in S'(\mathbb{R})$, $\mathrm{supp}(u_k) \subset \overline{\mathbb{R}^+}$, $k = 1,\dots,n$. *Then* $u_1 * \dots * u_n \in S'(\mathbb{R})$.

<u>PROOF</u>. $u_1 * \dots * u_n \in \mathcal{D}'(\mathbb{R})$ is welldefined and associative because all u_k have support in $\overline{\mathbb{R}^+}$. See section 2.2. Therefore we can assume $n = 2$. Choose $\chi \in C^\infty(\mathbb{R})$ so that

$$\chi(s) = \begin{cases} 1 & \\ & \text{for } s \\ 0 & \end{cases} \begin{matrix} > -\frac{1}{2} \\ \\ < -1 \end{matrix}.$$

Then for $\varphi \in C_0^\infty(\mathbb{R})$:

$$\text{(A.7.4)} \qquad \langle u_1 * u_2, \varphi\rangle = \langle u_1 \otimes u_2, \chi(x)\chi(y)\varphi(x+y)\rangle.$$

Let us show that this defines a continuous linear form on $S(\mathbb{R})$, too. Since $u_1 \otimes u_2 \in S'(\mathbb{R}^2)$ we must show first that $\chi(x)\chi(y)\varphi(x+y) \in S(\mathbb{R}^2)$. If $x^2 + y^2 = R^2$ then $x^2 \geq R^2/2$ or $y^2 \geq R^2/2$, so $|x| \geq R/\sqrt{2}$ or $|y| \geq R/\sqrt{2}$. We can assume $x \geq -1$ and $y \geq -1$, so if $R > \sqrt{2}$: $x+y \geq R/\sqrt{2} - 1$. But then we get for $\sqrt{x^2+y^2} \geq 2\sqrt{2}$, $x \geq -1$, $y \geq -1$ the inequalities:

$$\frac{1}{2\sqrt{2}}\sqrt{x^2+y^2} \leq \frac{1}{\sqrt{2}}\sqrt{x^2+y^2} - 1 \leq x+y \leq 2\sqrt{x^2+y^2}.$$

Now $\varphi \in S(\mathbb{R})$ implies

$$\forall (\ell,m): \exists M_{\ell,m}: \forall z: |z^\ell\varphi^{(m)}(z)| \leq M_{\ell,m} < \infty.$$

Also

$$|D^\alpha\chi(x)\chi(y)\varphi(x+y)| \leq \sum_{k=0}^{|\alpha|} C_k|\varphi^{(k)}(x+y)|, \quad C_k < \infty \text{ and independent of } \varphi.$$

But then for $\sqrt{x^2+y^2} \geq 2\sqrt{2}$:

$$|(x^2+y^2)^{\ell/2}D^{\alpha}\chi(x)\chi(y)\varphi(x+y)| \le (2\sqrt{2})^{\ell}(x+y)^{\ell}\sum_{k=0}^{|\alpha|} C_k|\varphi^{(k)}(x+y)|$$
$$\le (2\sqrt{2})^{\ell}\sum_{k=0}^{|\alpha|} C_k M_{\ell,k}.$$

For $\sqrt{x^2+y^2} \le 2\sqrt{2}$:

$$|(x^2+y^2)^{\ell/2}D^{\alpha}\chi(x)\chi(y)\varphi(x+y)| \le (2\sqrt{2})^{\ell}\sum_{k=0}^{|\alpha|} C_k M_{0,k}.$$

This implies $\chi(x)\chi(y)\varphi(x+y) \in S(\mathbb{R}^2)$. Moreover, if $\varphi_j \to 0$ in $S(\mathbb{R})$ then $\forall\ (\ell,m)$: $M^j_{\ell,m} \to 0$. So $\chi(x)\chi(y)\varphi_j(x+y) \to 0$ in $S(\mathbb{R}^2)$ as well because the C_k are independent of φ. Therefore expression (A.7.4) can be extended continuously to $S(\mathbb{R})$, that is, $u_1 * u_2 \in S'(\mathbb{R})$. □

REMARK. $u \in S'(\mathbb{R}^n) \Leftrightarrow$
$u = D_x^{\alpha}(1+|x|^2)^k f(x)$ for some α,k and bounded continuous function f. Further $(1+|x|^2)^k(1+|y|^2)^{\ell} \le (1+|x|^2+|y|^2)^{k+\ell}$, so $u \in S'(\mathbb{R}^n)$, $v \in S'(\mathbb{R}^m)$, implies $u \otimes v = D_x^{\alpha}D_y^{\beta}(1+|x|^2+|y|^2)^{k+\ell}h(x,y)$ with

$$h(x,y) = f(x)g(y)\frac{(1+|x|^2)^k(1+|y|^2)^{\ell}}{(1+|x|^2+|y|^2)^{k+\ell}}$$ a bounded continuous function.

So $u \otimes v \in S'(\mathbb{R}^{n+m})$.

LEMMA A.7.5. *Let* $a \in E'(\mathbb{R}^n_x)$ *and* $b \in \mathcal{D}'(\mathbb{R}^n_x \times \mathbb{R}^m_y)$. *Convolution of* a *and* b *with respect to* x *only is then welldefined by*

$$a *_x b := (a \otimes \delta_{y=0}) * b$$

and we have

$$WF(a *_x b) \subset \Big\{(x_1+x_2,y,\xi,\eta) \mid (x_2,y,\xi,\eta) \in WF(b) \text{ and } \big[(x_1,\xi) \in WF(a) \text{ or } (\xi=0 \text{ and } x_1 \in \operatorname{supp}(a))\big]\Big\}.$$

PROOF. $a \otimes \delta_{y=0} \in E'(\mathbb{R}^n_x \times \mathbb{R}^m_y)$ so convolution is welldefined.

$$WF(a\otimes\delta_{y=0}) \overset{(1)}{\subset} WF(a)\times WF(\delta_{y=0}) \cup ((\operatorname{supp}(a)\times\{0\})\times WF(\delta_{y=0})) \cup (WF(a)\times(\operatorname{supp}(\delta_{y=0})\times\{0\}))$$
$$= \Big\{(x,0,\xi,\eta) \mid (x,\xi)\in WF(a) \text{ or } (\xi=0, \eta\ne 0 \text{ and } x\in\operatorname{supp}(a))\Big\}.$$

Therefore

$$WF(a *_x b) = WF((a\otimes\delta_{y=0}) * b)$$
$$\overset{(2)}{\subset} \Big\{(x_1+x_2,y_1+y_2,\xi,\eta) \mid (x_1,y_1,\xi,\eta) \in WF(a\otimes\delta_{y=0}) \text{ and}$$

$$(x_2,y_2,\xi,\eta) \in WF(b)\Big\}$$

$$\subset \Big\{(x_1+x_2,y,\xi,\eta) \mid (x_2,y,\xi,\eta) \in WF(b) \text{ and } \Big[(x_1,\xi) \in WF(a) \text{ or } (\xi = 0 \text{ and } x_1 \in \text{supp}(a))\Big]\Big\}.$$

Step (1) follows from property 5 in section 2.4, step (2) follows from the result given in section 2.5. □

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